TEST EDITION

THE TEXTBOOK OF

MATHEMATICS

For Class XII

SINDH TEXTBOOK BOARD, JAMSHORO

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PREFACE

It is with great pleasure and immense gratitude that we present to you the mathematics textbook for Class XII, developed under the guidance of the Sindh Textbook Board, Jamshoro. As an institution committed to shaping the educational landscape of the province, we take immense pride in offering a comprehensive and meticulously crafted resource that aligns with the dynamic curriculum set forth by the Directorate of Curriculum, Assessment, and Research (DCAR) in 2019.

This textbook is a collaborative effort that owes its existence to the dedication and expertise of countless individuals. We extend our sincere thanks to the authors, editors, and reviewers who tirelessly contributed their knowledge, insights, and time to create a textbook that not only imparts mathematical concepts but also fosters a love for learning and discovery.

The mathematics textbook for Class XII delves into the fascinating worlds of Calculus and Analytic Geometry. These subjects form the cornerstone of modern mathematics and have far-reaching applications in various fields. The content has been meticulously structured to provide students with a solid foundation in these areas, preparing them to tackle real-world challenges and pursue higher studies with confidence.

This textbook is designed with the utmost care to enhance students' understanding of mathematical concepts through clear explanations, illustrative examples, and thought-provoking exercises. It is our hope that this resource will serve as a valuable companion to both students and teachers, facilitating a deeper grasp of the subject matter and encouraging analytical thinking.

We believe that education is a continuous journey, and as such, we welcome your feedback and suggestions. Our commitment to improvement and innovation remains unwavering, and we encourage students and educators alike to share their thoughts and ideas. Together, we can create an educational ecosystem that empowers learners to excel and educators to inspire.

In closing, we extend our heartfelt wishes for your academic journey with this textbook. May it instill in you a passion for learning, a thirst for knowledge, and a profound appreciation for the beauty of mathematics.

Chairman

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Functions and Limits

Unit 2

2.1 Functions

We know that function is a rule or correspondence between two non-empty sets X and Y in such a way that, each element of X corresponds to one and only one element of Y. Here X is called the domain of the function and the set of corresponding elements of Y is called the range of the function.

2.1.1 Identify through graph the domain and range of a function

Graph of a function is useful to identify the domain and range of a function. The domain of the function consists of all the input values shown on the x-axis. The range is the set of possible output values shown on the y-axis.

For example, consider the graph of function as shown in Fig. 2.1. We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $\{x|x \in \mathbb{R} \wedge x \geq -5\}$. The graph extends vertically from 5 to downward without bound, so the range of the function is $\{y|y \in \mathbb{R} \wedge y \leq 5\}$.

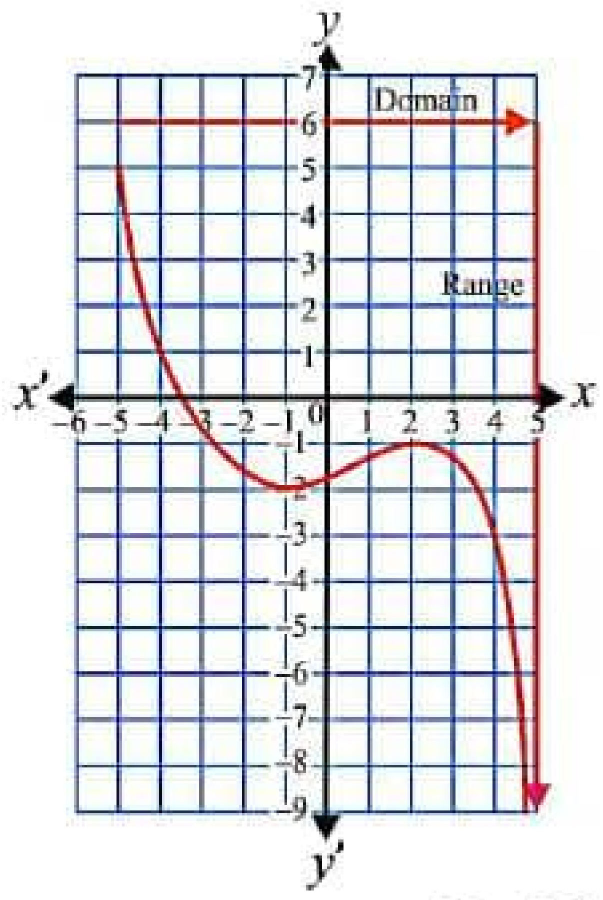


Fig. 2.1

Example 1. Identify the domain and range of the function through given graph.

Solution: The given graph is shown in Fig. 2.2.

We can observe that the graph extends horizontally from -3 to 1. So, the domain is $\{x|x \in \mathbb{R} \wedge -3 \leq x \leq 1\}$.

The graph extends vertically from 0 to -4. So, the range is $\{y|y \in \mathbb{R} \wedge -4 \leq y \leq 0\}$.

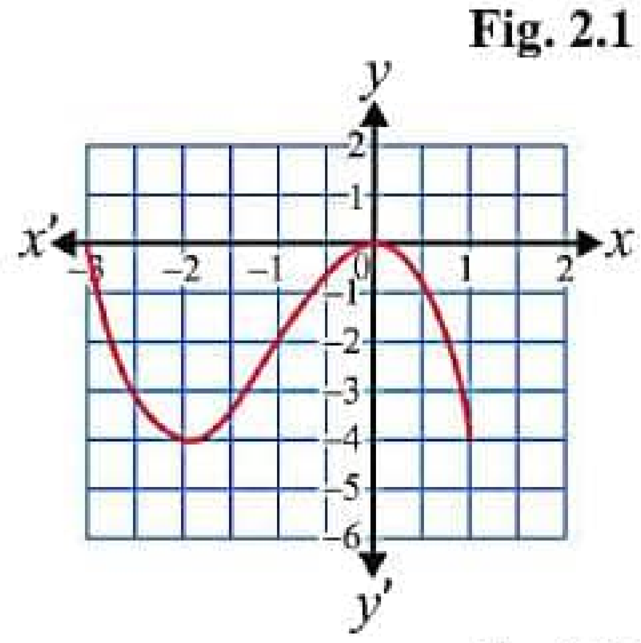


Fig. 2.2

Example 2. Identify the domain and range of the function through given graph.

Solution: The given graph is shown in Fig. 2.3.

The graph extends horizontally and vertically without and any bound. Thus, the domain and range of the function is $\{\mathbb{R}\}$. (Fig. 2.3)

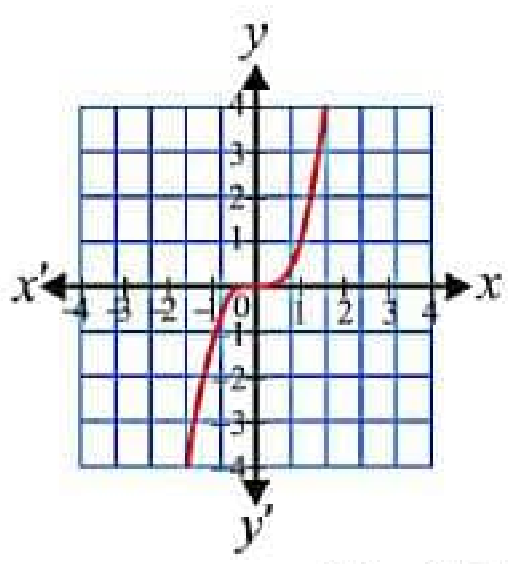


Fig. 2.3

2.1.2 Draw the graph of modulus function (i.e., $y = |x|$) and identify its domain and range

The modulus function $y = |x|$ is defined as $|x| = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$.

First, we draw the graph with the help of following table.

x	0	1	−1	2	−2	3	−3	...
$y = \|x\|$	0	1	1	2	2	3	3	...

By plotting these points on coordinate axes. We get, the graph of modulus function (Fig. 2.4). Now, we identify its domain and range with the help of graph.

The arrows indicate that the graph extends horizontally without any bound, so the domain is $\mathbb{R}$. While, the graph extends vertically from 0 to upward without any bound. So, its range is $\{y | y \in \mathbb{R} \land y \geq 0\}$.

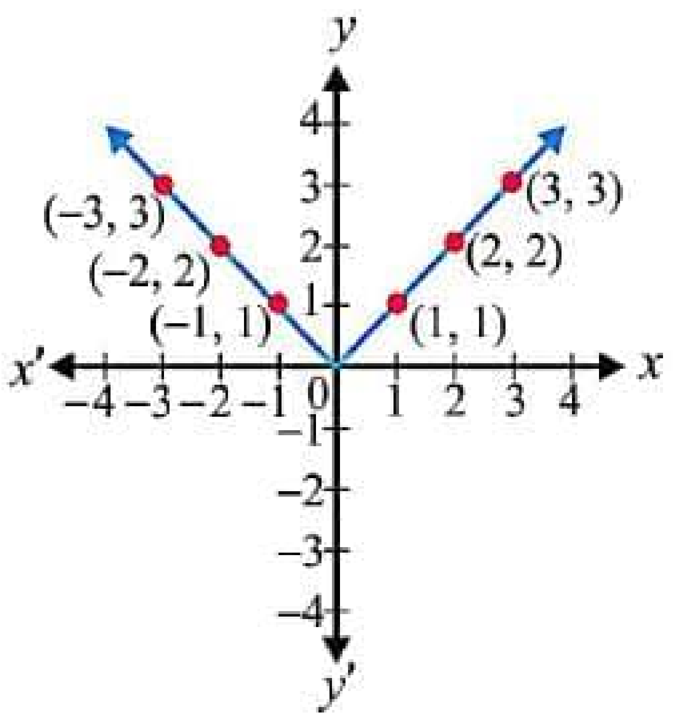


Fig. 2.4

2.2 Composition of Functions

Composition of functions is an operation or process where two functions f and g produce a new function h by replacing the variable of one function with other function.

2.2.1 Recognize the composition of functions

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Then the composition of f and g, denoted by $\boldsymbol{gof}$, is defined as the function

$gof: A \rightarrow C$, given by $gof(x) = g(f(x)), \forall\, x \in A$.

The composition gof of functions f and g exists when Range $f \subseteq$ domain of g. The domain and range of composite function gof will be domain of f and range of g respectively as shown in the Fig. 2.5.

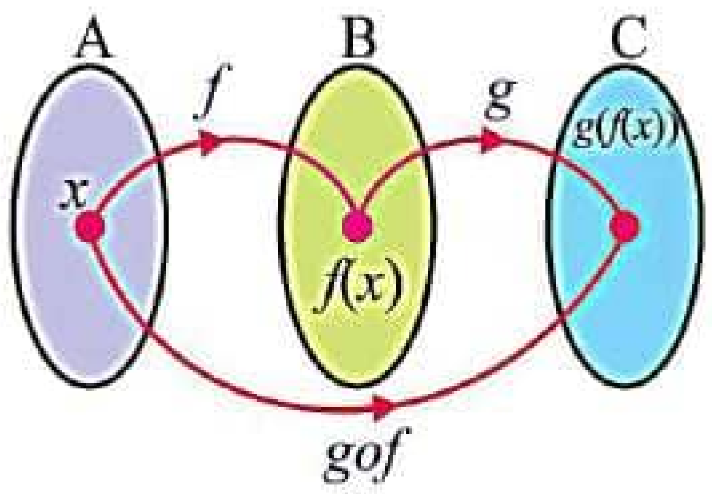


Fig. 2.5

The order of function is an important while dealing with the composition of functions since $gof(x)$ is not equal to $fog(x)$ in general.

2.2.2 Find the composition of two given functions

Example 1. If $f: R \longrightarrow R$ is a function which is defined as $f(x) = 3x + 1$ and $g: R \longrightarrow R$ is another function which is defined as $g(x) = x^2$. Find $fog(x)$.

Solution: Since range $g = R \subseteq$ Domain f, therefore fog exists.

The composition of f and g will be

$$fog(x) = f\big(g(x)\big) = f(x^2)$$
$$= 3(x^2) + 1.$$
$$fog(x) = 3x^2 + 1.$$

Example 2. $f(x) = 2x + 1$ and $g(x) = -x^2$, then find $gof(x)$ for $x = 2$.

Solution:

The composition of g and f will be

$$gof(x) = g\big(f(x)\big) = g(2x + 1)$$
$$= -(2x + 1)^2$$

Now,

$$gof(2) = -[2(2) + 1]^2$$
$$= -(5)^2$$
$$= -25$$

Example 3. If $R \longrightarrow [-1, 1]$ is sine function i.e., $s(x) = \sin x$ and $p(x)$ is a polynomial function i.e., $p(x) = x^2 + 5x + 7$ then find pos.

Solution:

$$pos(x) = p\big(s(x)\big)$$
$$= p(\sin x)$$
$$= (\sin x)^2 + 5(\sin x) + 7$$
$$= \sin^2 x + 5\sin x + 7$$

2.3 Inverse of Composition of Functions

2.3.1 Describe the inverse of composition of two given functions

Let f and g are bijective functions then inverse of composition of f and g is the composition of g^{-1} and f^{-1}. Mathematically, $(fog)^{-1} = g^{-1}of^{-1}$.

Example: If $f(x) = \frac{x+1}{2}$ and $g(x) = 2x - 1$ are two given bijective functions then find the inverse of composition of f and g, also show that $(gof)^{-1} = f^{-1}og^{-1}$

Solution:

Here $f(x) = \frac{x+1}{2}$ and $g(x) = 2x - 1$

Now, we find $gof(x) = g\big(f(x)\big)$

$$= g\left(\frac{x+1}{2}\right)$$
$$gof(x) = 2\left(\frac{x+1}{2}\right) - 1 = x$$

$\Rightarrow \quad (gof)^{-1} = x$

Now, we verify $(gof)^{-1} = f^{-1}og^{-1}$

$$f(x) = \frac{x+1}{2}$$

$\Rightarrow \quad f^{-1}(x) = 2x - 1$

and $\quad g(x) = 2x - 1$

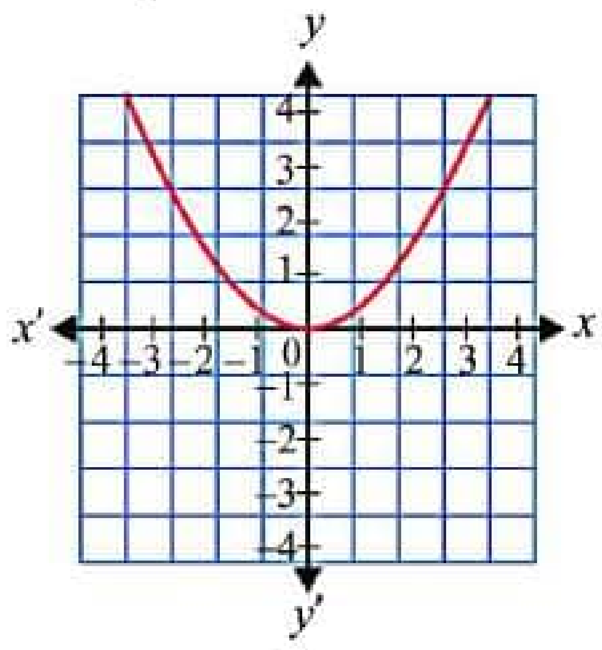
$\Rightarrow \quad g^{-1}(x) = \dfrac{x+1}{2}$

$$f^{-1}og^{-1} = f^{-1}(g^{-1}(x)) = f^{-1}\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x$$

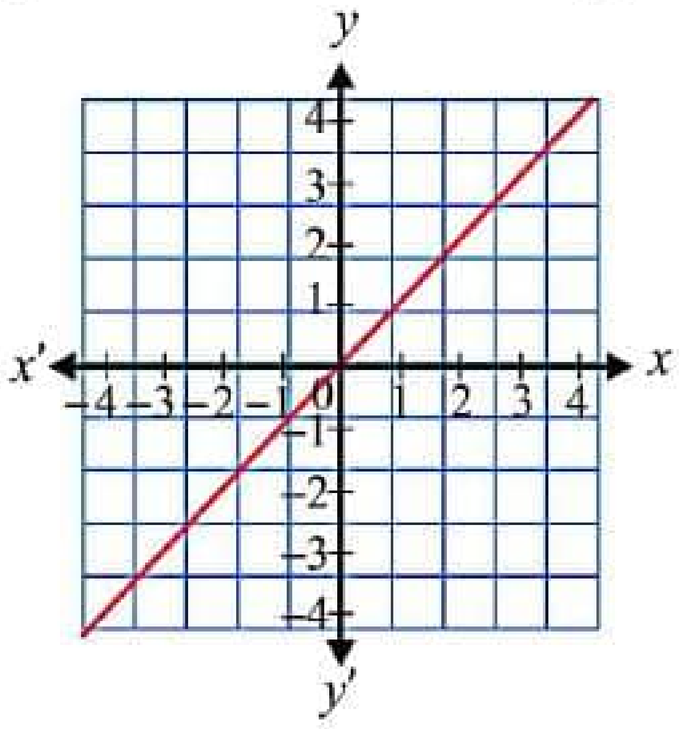
Hence $f^{-1}og^{-1} = (gof)^{-1}$ shown.

Exercise 2.1

1. Identify the domain and range of the functions through following graph.



(i)



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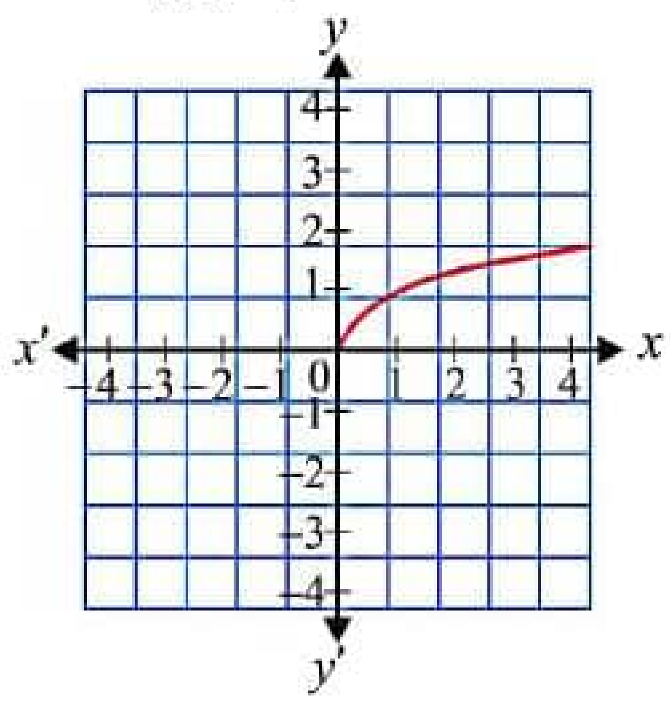

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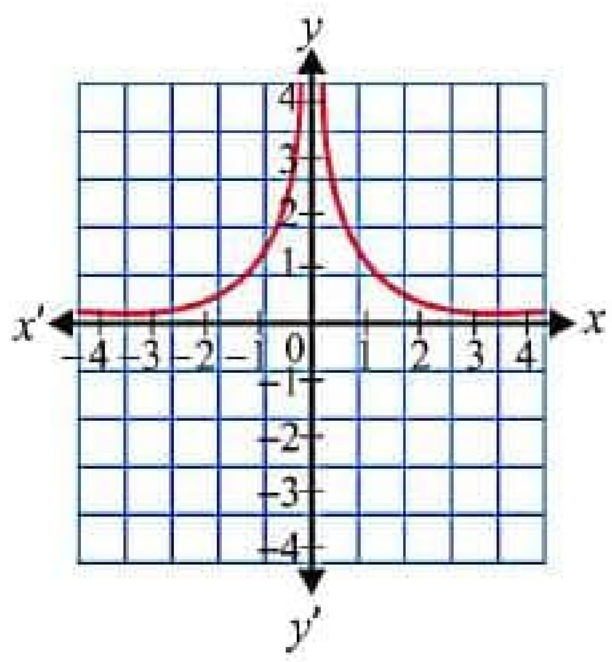

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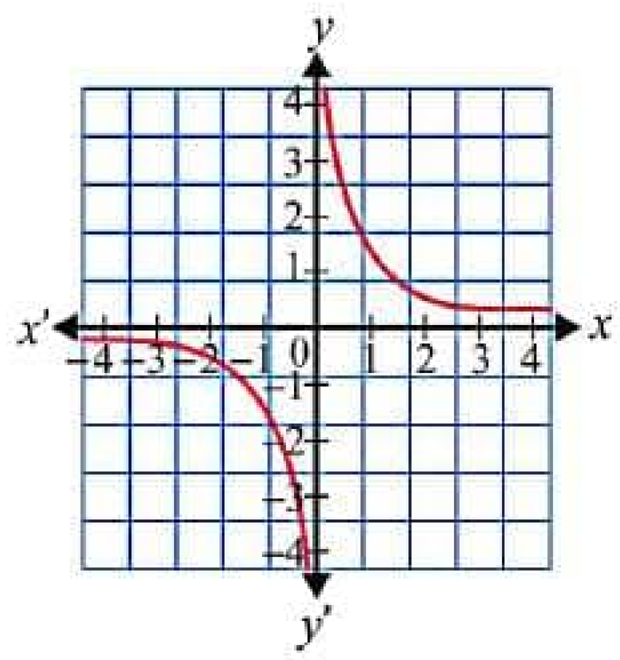


(v)

2. If $f(x) = 5x + 2$ and $g(x) = 2x^2 - 3$, then find

   (i) fog (ii) gof (iii) fof (iv) gog

3. If $f(x) = 2x$ and $g(x) = x + 1$, then find $fog(x)$ for $x = -5$.

4. If $f(x) = x + 3$ and $g(x) = x^2$, then find $gof(x)$ for $x = 1$.

5. If $c(x) = \cos x$ and $p(x) = x^3 + 1$ then find $poc(x)$.

6. Given that $f(x) = x + 2$ and $g(x) = 3x - 2$ are two given functions then find $(fog)^{-1}$ and $(gof)^{-1}$ also show that $(gof)^{-1} = f^{-1}og^{-1}$.

7. Given that $h(x) = x - 3$ and $k(x) = 2x + 5$ are two functions then verify that:

   (i) $hok \neq koh$ (ii) $(hok)^{-1} = k^{-1}oh^{-1}$ (iii) $(koh)^{-1} = h^{-1}ok^{-1}$

2.4 Transcendental Functions

All the functions other than algebraic functions are transcendental functions. Like, $\sin x$, $\cos^{-1} x$, $\ln x$, e^x and sinh$x$ etc.

2.4.1 Recognize algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions.

Some important types of functions are as under:

(a) Algebraic functions

(b) Transcendental functions

(c) Explicit and Implicit functions

(d) Parametric functions

(a) Algebraic functions:

Algebraic function is a function which is defined by algebraic expression that contain only algebraic operations. For example, $p(x) = x^2 + 5x + 7, q(x) = \frac{x+1}{x-2} + 7$ and $r(x) = \sqrt{x+1} + 8x^2 + 9$ are algebraic function.

(b) Transcendental functions: Exponential functions, logarithmic functions, trigonometric functions, hyperbolic functions, and inverse of all these functions are called transcendental functions.

(I) Exponential functions: If $f(x) = a^x$ where $a \in R^+$ and $a \neq 1$ then $f(x)$ is called an exponential function of x to the base a. For example, $f(x) = 3^x$; $f(x) = \left(\frac{1}{2}\right)^x$; $h(x) = \left(\sqrt{5}\right)^x$ and $k(x) = (7)^{-x}$ are exponential functions.

The function e^x is called the natural exponential function where $e = 2.718281 \ldots$

(II) Logarithmic function: If $y = a^x$ where $a \in R^+$ and $a \neq 1$ then $\log_a y = x$ is called logarithmic function of y to the base a.

Note:

(i) $y = \log_{10} x$ is a Logarithmic function of base 10 which is called common logarithmic function.

(ii) $y = \log_e x$ or $y = \ln x$ is a Logarithmic function of base e which is called natural logarithmic function.

(iii) The Logarithmic function is the inverse of the exponential function.

(iv) The domain of Logarithmic function is $\mathbb{R}^+$ and its range is $\mathbb{R}$.

(v) $a^y = e^{y \cdot \ln a}$ (vi) $\log_a a = 1$; $\log_a 1 = 0$

(III) Trigonometric functions: We have already studied the six trigonometric functions in previous classes which are $\sin x, \cos x, \tan x, \sec x, \csc x$ and $\cot x$ are the trigonometric functions.

(IV) Inverse trigonometric functions: We have already studied the inverse trigonometric function in previous class. $\text{Sin}^{-1} x, \text{Cos}^{-1} x, \text{Tan}^{-1} x, \text{Sec}^{-1} x, \text{Csc}^{-1} x$ and $\text{Cot}^{-1} x$ are the inverse trigonometric functions.

(V) Hyperbolic functions: Hyperbolic functions are defined in a way similar to trigonometric functions. As the name suggests, the graph of a hyperbolic function represents a hyperbola. They are expressed in terms of exponential function e^x. There are six hyperbolic functions which are defined as under:

(i) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

is called sine hyperbolic function of x its domain and range are $\mathbb{R}$.

(ii) $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

is called cosine hyperbolic function of x its domain is $\mathbb{R}$ and range is $\{y | y \in \mathbb{R} \land y \geq 1\}$.

(iii) $y = \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

is called tangent hyperbolic function, its domain is $\mathbb{R}$ and range is $\{y | y \in \mathbb{R} \land -1 \leq y \leq 1\}$.

(iv) $y = \text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

is called secant hyperbolic function of x its domain is $\mathbb{R}$ and range $\{y | y \in \mathbb{R} \land 0 < y \leq 1\}$.

(v) $y = \text{csch}\, x = \dfrac{1}{sinh\, x} = \dfrac{2}{e^x - e^{-x}}$

is called cosecant hyperbolic function of x, its domain and range $\{y | y \in \mathbb{R} \land y \neq 0\}$.

(vi) $y = \coth x = \dfrac{1}{\tanh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

is called cotangent hyperbolic function of x, its domain is $\{x | x \in \mathbb{R} \land x \neq 0\}$ and range is $\{y | y \in \mathbb{R} \land y \leq -1 \land y \geq 1\}$.

(VI) Inverse Hyperbolic functions:

The inverse hyperbolic functions are defined as under:

(i) $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ is inverse sine hyperbolic function its domain and range is $\mathbb{R}$.

(ii) $y = \cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$, is inverse cosine hyperbolic function, its domain is $\{x | x \in \mathbb{R} \wedge x > 1\}$ and range is $\{y | y \in \mathbb{R} \wedge y \geq 0\}$.

(iii) $y = \tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ is inverse tangent hyperbolic function, its domain is $\{x | x \in \mathbb{R} \wedge -1 < x < 1\}$ and range is $\mathbb{R}$.

(iv) $y = \text{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ is inverse secant hyperbolic function, its domain is $\{x | x \in \mathbb{R} \wedge 0 < x \leq 1\}$ and range is $\{y | y \in \mathbb{R} \wedge y \geq 0\}$.

(v) $y = \text{csch}^{-1}x = \text{In}\left(\frac{1}{x} + \frac{1+\sqrt{1+x^2}}{|x|}\right)$ is inverse cosecant hyperbolic function, its domain and range are $\{y | y \in \mathbb{R}, y \neq 0\}$.

(vi) $y = \coth^{-1}x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|$

is inverse cotangent hyperbolic functions. Its domain is $\{x | x \in \mathbb{R} \wedge x \neq 1\}$ and range is $\{y | y \in \mathbb{R} \wedge y \neq 0\}$.

Identities of trigonometric and hyperbolic functions

	Trigonometric Identities	Hyperbolic Identities
i.	$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
ii.	$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \text{sech}^2 x$
iii.	$1 + \cot^2 x = \text{cosec}^2 x$	$\coth^2 x - 1 = \text{cosech}^2 x$
iv.	$\sin 2x = 2\sin x\cos x$	$\sinh 2x = 2\sinh x\cosh x$
v.	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
vi.	$\cos 2x = 2\cos^2 x - 1$	$\cosh 2x = 2\cosh^2 x - 1$
vii.	$\cos 2x = 1 - 2\sin^2 x$	$\cosh 2x = 2\sinh^2 x + 1$
viii.	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
ix.	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
x.	$\sin(x \pm y)$ $= \sin x\cos y \pm \cos x\sin y$	$\sin h(x \pm y)$ $= \sinh x\cosh y \pm \cosh x\sinh y$
xi.	$\cos(x \pm y)$ $= \cos x\cos y \mp \sin x\sin y$	$\cosh(x \pm y)$ $= \cosh x\cosh y \pm \sinh x\sinh y$
xii.	$\sin(-x) = -\sin x$	$\sinh(-x) = -\sinh x$
xiii.	$\cos(-x) = \cos x$	$\cosh(-x) = \cosh x$

(c) **Explicit and Implicit functions:**

(I) **Explicit function:** An explicit function is a function in which dependent variable y can be written explicitly only in terms of the independent variable x. Mathematically, it is written as $y = f(x)$. For example, $y = x - 1, y = e^x + \sin x$ etc.

(II) **Implicit function:** A function in which dependent variable y can not be expressed explicitly in terms of independent variable x. Both dependent variable y and independent variable x are mixed with each other where y cannot be expressed isolately as the function of x.

For example, $x^2 + xy + y^2 = 0$, where y is the implicit function of x.

(d) **Parametric Representation of Function:** A function can be represented parametrically by expressing the both dependent and independent variable as the functions of parameter such as t.

For example, $x = \cos t$ and $y = \sin t$ are the parametric representation of $x^2 + y^2 = 1$, here t is a parameter.

2.5 Graphical Representations

2.5.1 Display graphically:

The explicitly defined functions like $y = f(x)$, where

- $f(x) = e^x$,
- $f(x) = a^x$,
- $f(x) = log_a x$
- $f(x) = log_e x$

(i) $f(x) = e^x$

The graph of $y = e^x \; \forall \; x \in \mathbb{R}$.

It is observed from the graph of e^x, as shown in Fig. 2.6, the function e^x is increasing and its graph cuts the y-axis at $(0, 1)$.

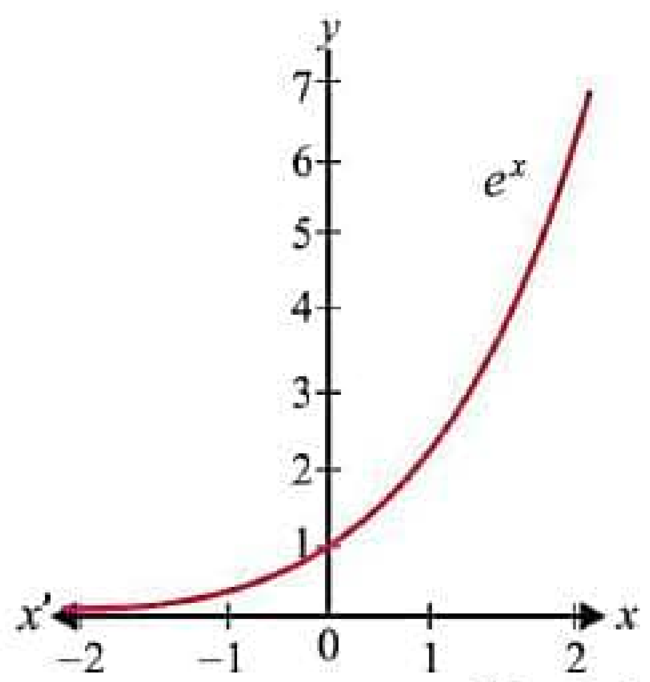


Fig. 2.6

(ii) $f(x) = a^x$

The graph of the function $f(x) = a^x$ is similar to the graph of e^x and its shape depends on the changing value of 'a'. The graph of $f(x) = a^x$ for different value of 'a' is show in Fig. 2.7.

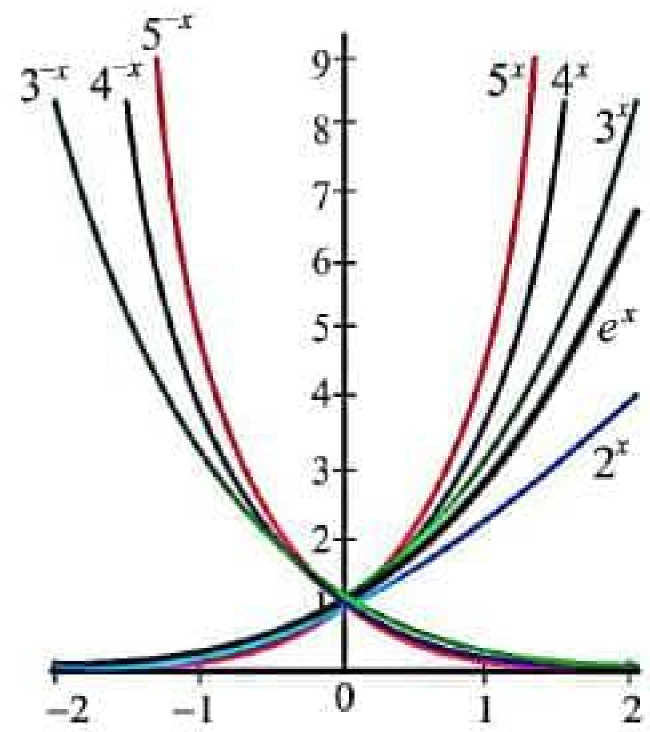


Fig. 2.7

The graph indicates that:

- If $0 < a < 1$ then $y = a^x$ is decreasing function.
- If $a > 1$ then $y = a^x$ is increasing function.

- The curves $y = a^x$ close to the positive y-axis as $a > 0$ increases.

(iii) $f(x) = log_a x$

A function of the form $y = f(x) = log_a x$ where $a > 1$ is known as logarithmic function with base $'a'$. Its graph depends on the value of $'a'$.

Since the inverse function of $y = log_a x$ is the exponential function $y = a^x$, the graph of $y = log_a x$ is the reflection of $y = a^x$ with the line $y = x$ as shown in the Fig. 2.8.

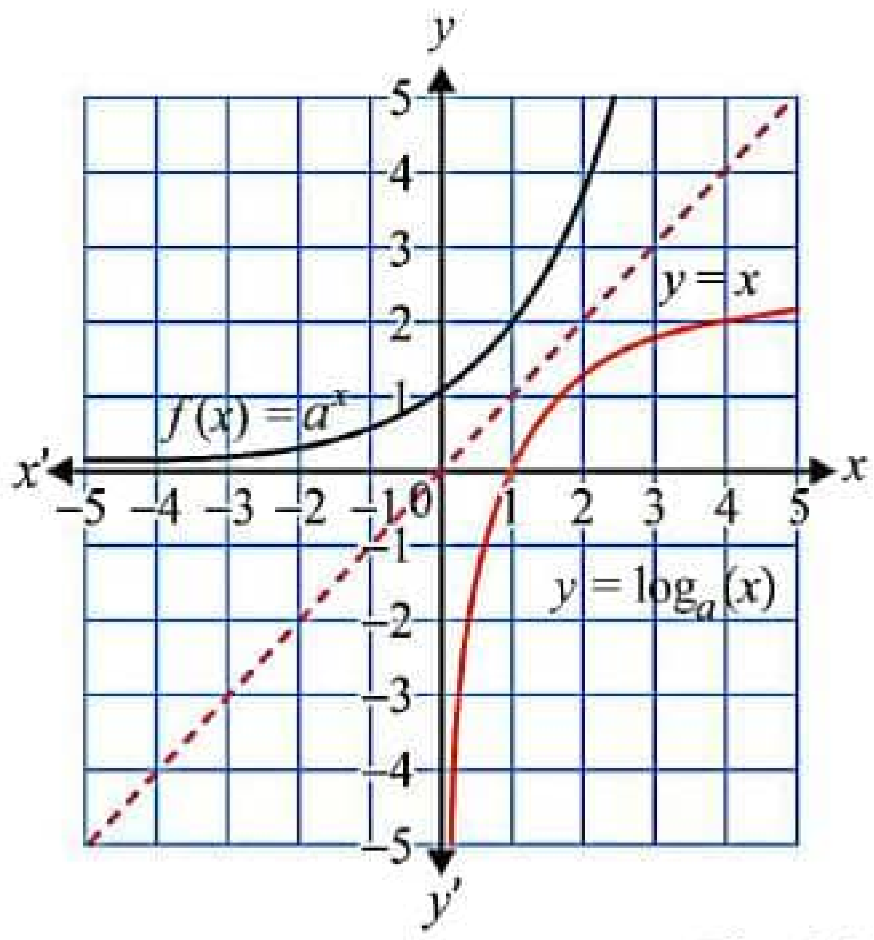


Fig. 2.8

The graph indicates that:

- The curve $y = log_a x$ cuts the x-axis at the point (1,0).
- the domain of the curve $y = log_a x$ is $\mathbb{R}$.
- the curves $y = log_a x$ approaches to negative y-axis as $x \in (0,1)$ as shown in Fig. 2.29.

- Changing the base b in $f(x) = log_b x$ can affect the graphs of $f(x)$. It is observed that the graph compresses vertically as the value of the base increases.
- $f(x)$ increases if $b > 1$, Fig. (2.10)
- $f(x)$ decreases if $0 < b < 1$, Fig.(2.11)

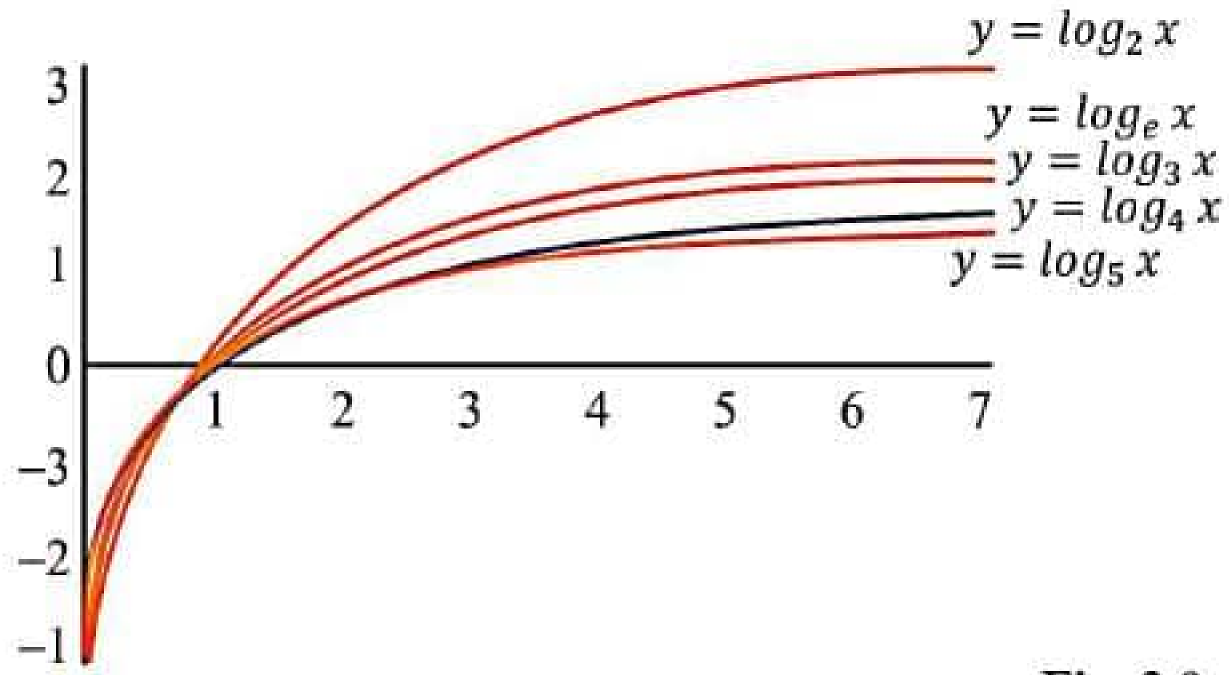


Fig. 2.9

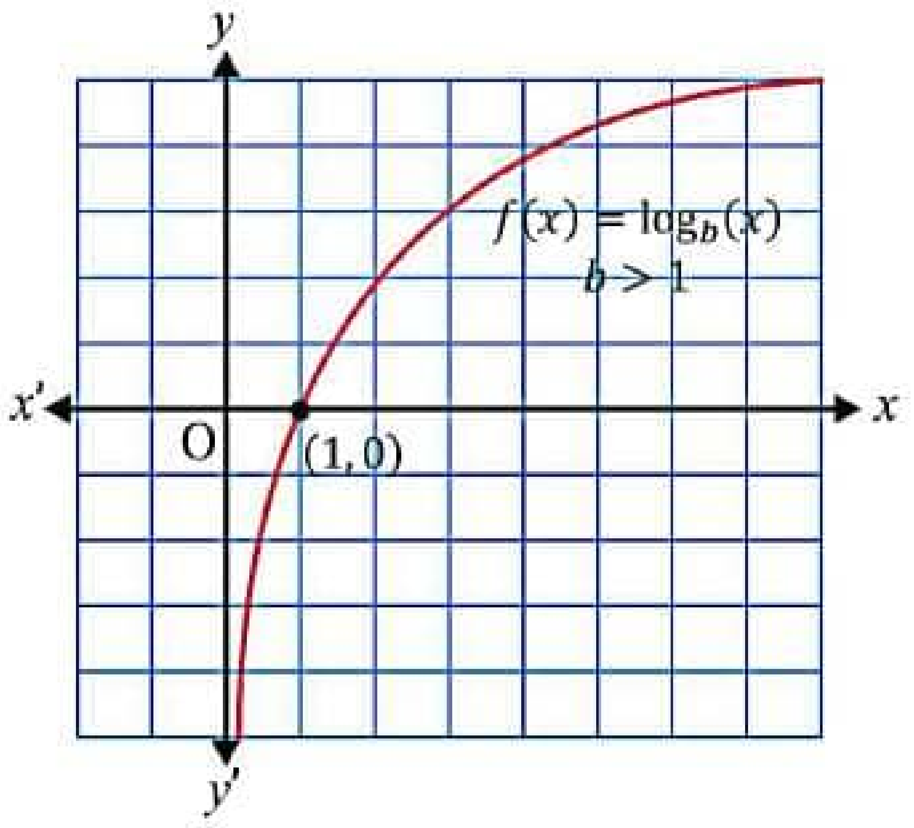


Fig. 2.10

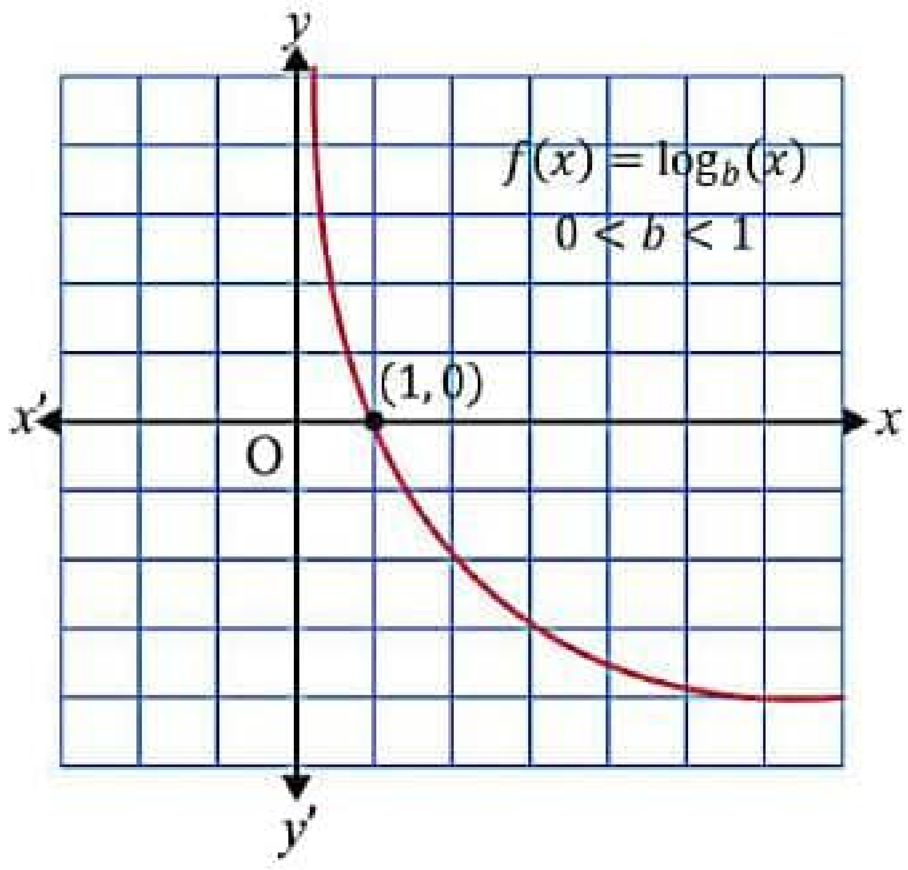


Fig. 2.11

(iv) $f(x) = log_e x$

Logarithmic function $f(x) = log_e x$ is known as natural Logarithmic function and represented by $\ln x$, which is displayed as in Fig. 2.12.

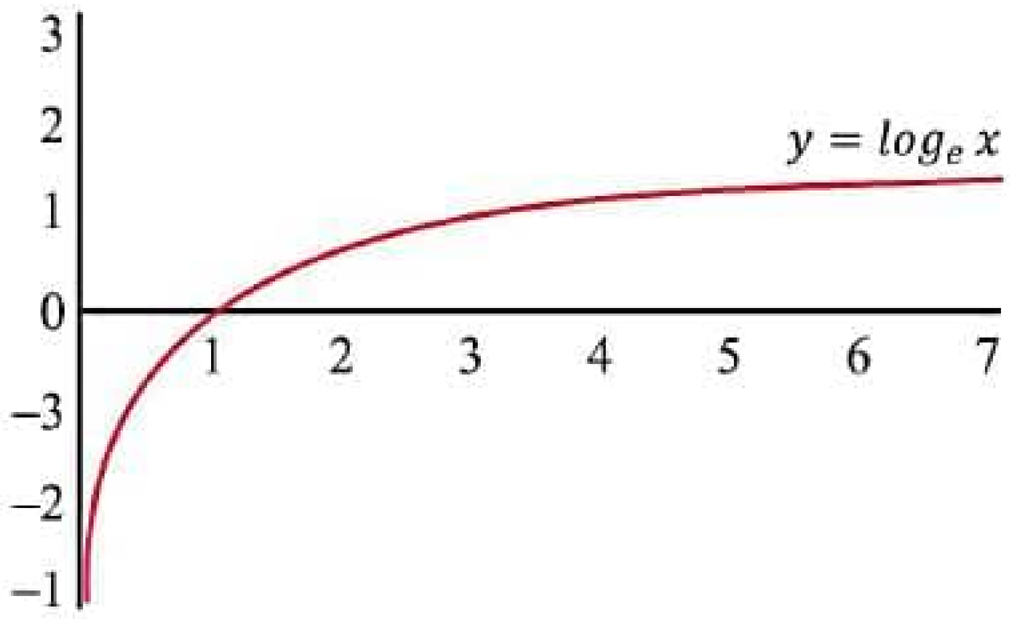


Fig. 2.12

- **Display graphically the implicitly defined functions such as $x^2 + y^2 = a^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and distinguish between graph of a function and an equation**

To display graphically the implicitly defined function, we solve the equation $f(x, y) = 0$ for y in terms of x where more than one function may be obtained.

Now, we draw the graphs of each function separately. Finally, by combining both graph, the graph of $F(x, y) = 0$ can be obtained.

For example, to display graphically $x^2 + y^2 = a^2$, first we solve y for x,

We get

$$y = \sqrt{a^2 - x^2} \text{ or } y = -\sqrt{a^2 - x^2}$$

Now, we separately draw the graph of each function as shown in Fig. 2.13 and 2.14.

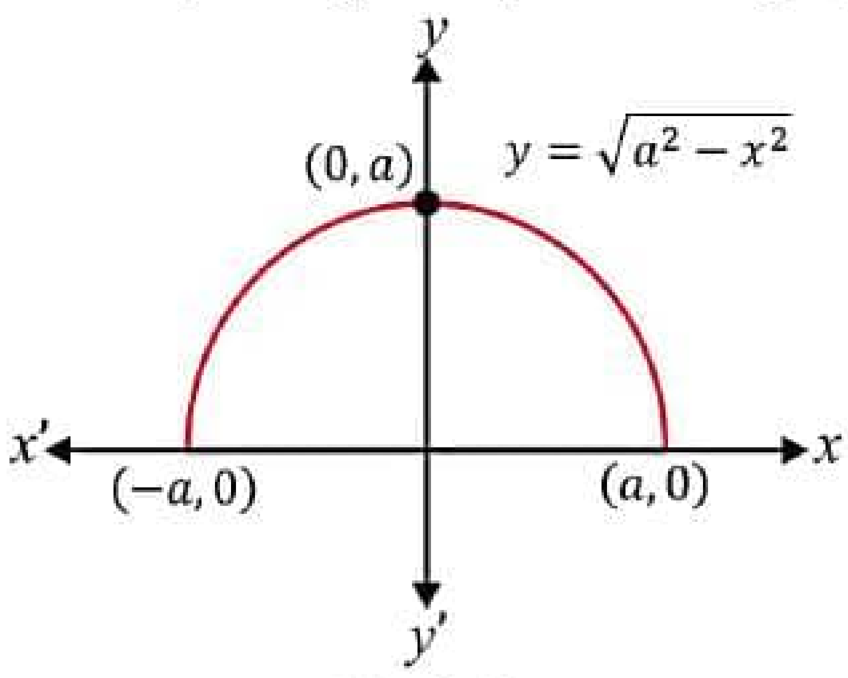


Fig. 2.13

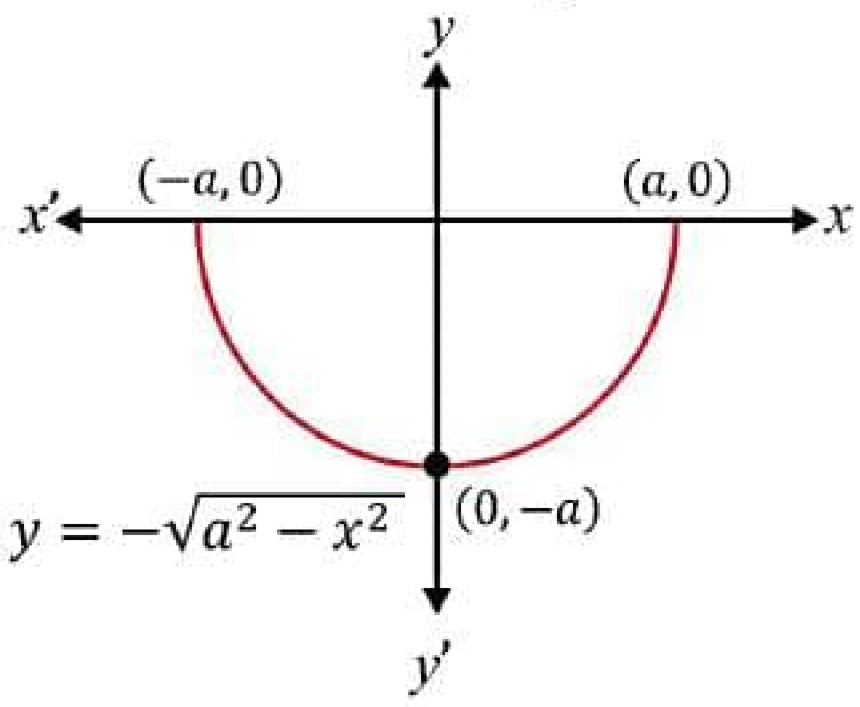


Fig. 2.14

Now, by combining both graphs, we get the graph of $x^2 + y^2 = a^2$ as shown in Fig. 2.15.

It is circle whose radius is 'a' unit and centre at the origin.

Similarly, we can display the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

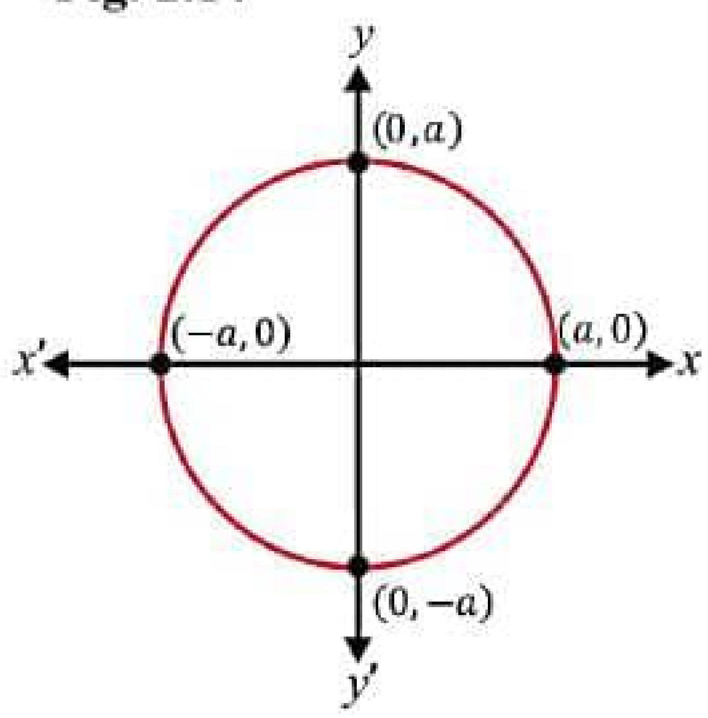


Fig. 2.15

It is an ellipse whose major axis is along x-axis and minor axis is along y-axis as shown in Fig 2.16.

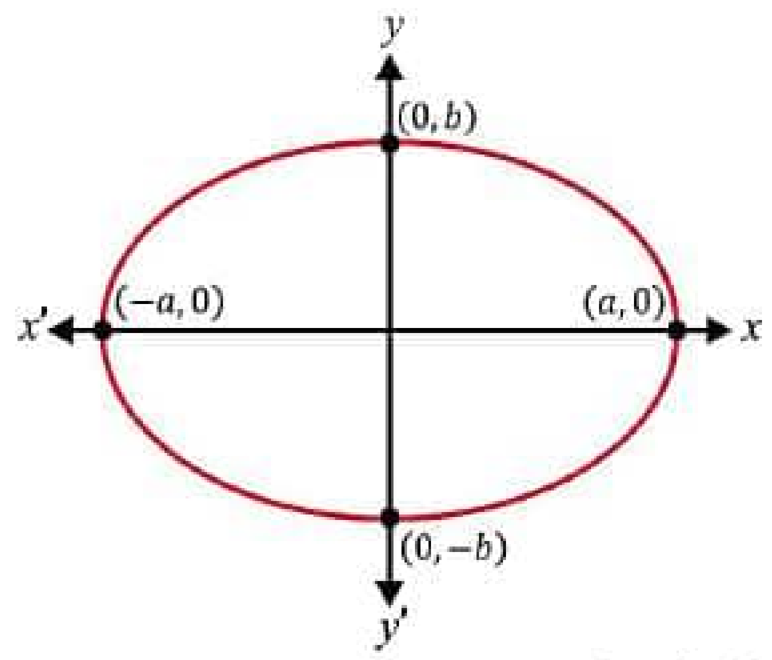


Fig. 2.16

Distinguish between graph of a function and an equation:

The graph of a function and an equation (implicitly defined function) can be distinguished by drawing vertical line on the same plane. If vertical line cuts the graph at only one point, then it is the graph of the function and if it cuts the graph at more than one point then it is the graph of the equation.

Example: Distinguish between the following graphs of function and equation.

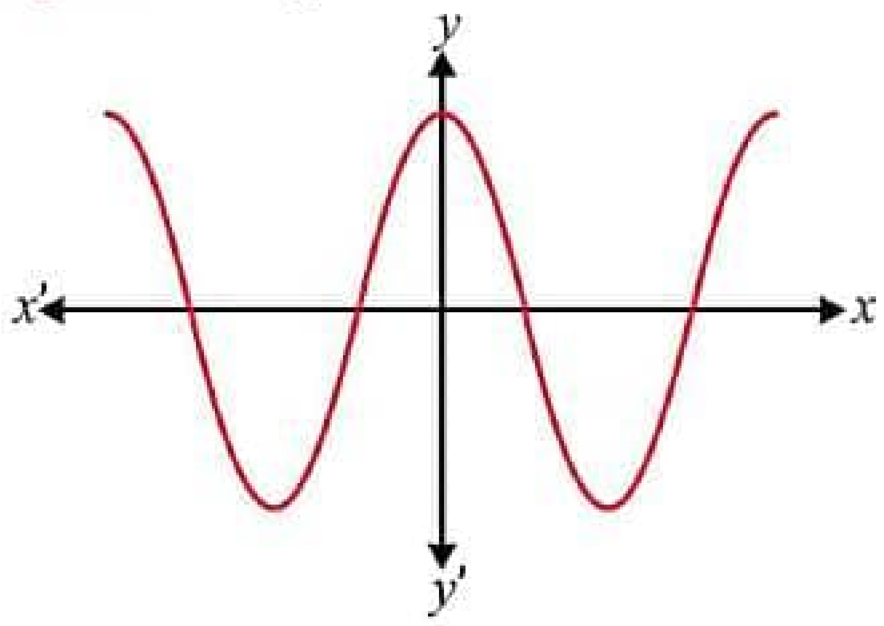


Fig. 2.17 (a)

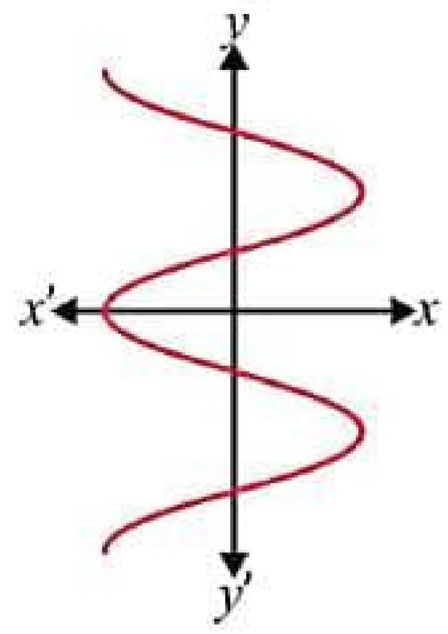


Fig. 2.17 (b)

We check it through vertical line test.

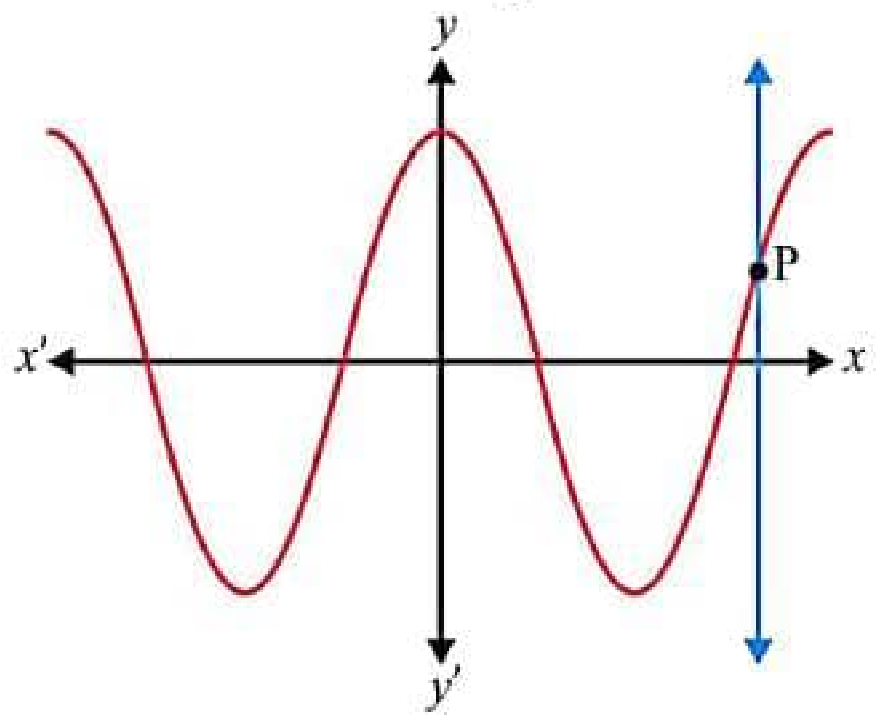


Fig. 2.18 (a)

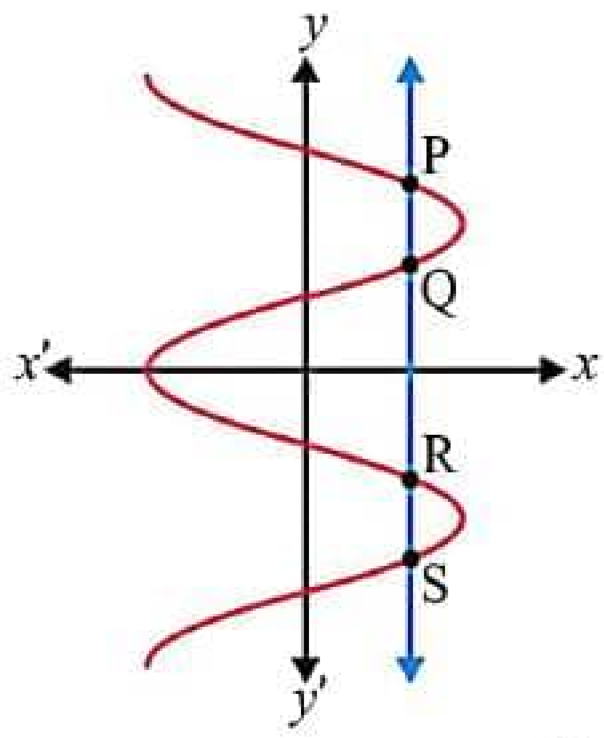


Fig. 2.18 (b)

In Fig. 2.18 (a) vertical line touches the graph at only one point P, so the Fig. 2.18 (a) represents the graph of the function.

In Fig. 2.18 (b) vertical line touches the graph at more than one point, so Fig. 2.18 (b) it is the graph of the equation.

- **Display graphically the parametric equations of functions such as $x = at^2,\ y = 2at\ ;\ x = a\sec\theta,\ y = b\tan\theta$**

Example 1. Draw the graph of parametric equations of function $x = at^2,\ y = 2at$, when $a = 2$ and $-3 \le t \le 3$

Solution: Parametric equations for $a = 2$ are

$$x = 2t^2 \quad \ldots\text{(i)}$$

$$y = 4t \quad \ldots\text{(ii)}$$

By constructing a table as $-3 \le t \le 3$

t	-3	-2	-1	0	1	2	3
$x = 2t^2$	18	8	2	0	2	8	18
$y = 4t$	-12	-8	-4	0	4	8	12

By plotting the points (x, y) on coordinate axes, we get the required graph Fig. 2.19.

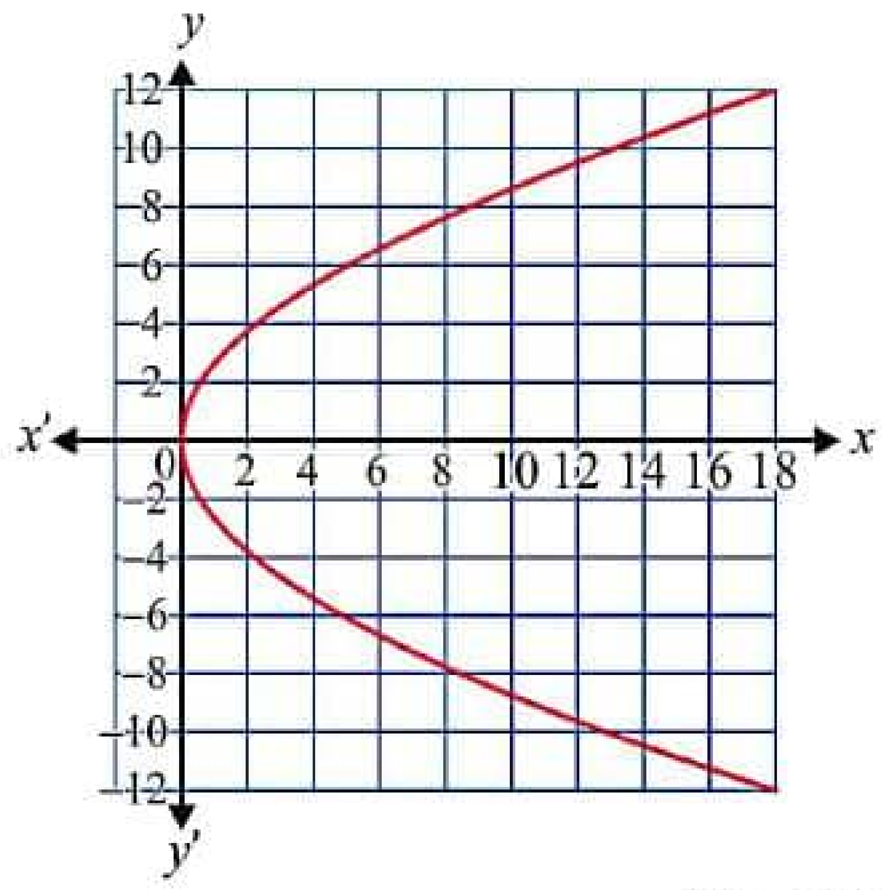


Fig. 2.19

Example 2. Draw the graph of parametric equations of function $x = a\sec\theta,\ y = b\tan\theta$, where $a = 5,\ b = 3$ and $-\pi \le \theta \le \pi$.

Solution:

Parametric equations for $a = 5$ and $b = 3$ are

$$x = 5\sec\theta \quad \ldots\text{(i)}$$

$$y = 3\tan\theta \quad \ldots\text{(ii)}$$

By constructing a table as $-\pi \le \theta \le \pi$

θ	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = 5\sec\theta$	-5	-5.8	-10	∞	10	5.8	5	5.8	10	∞	-10	-5.8	-5
$y = 3\tan\theta$	0	1.7	5.2	$-\infty$	-5.2	-1.7	0	1.7	5.2	∞	-5.2	-1.7	0

By plotting the points (x, y) on coordinate axes, we get the required graph of Fig. 2.20.

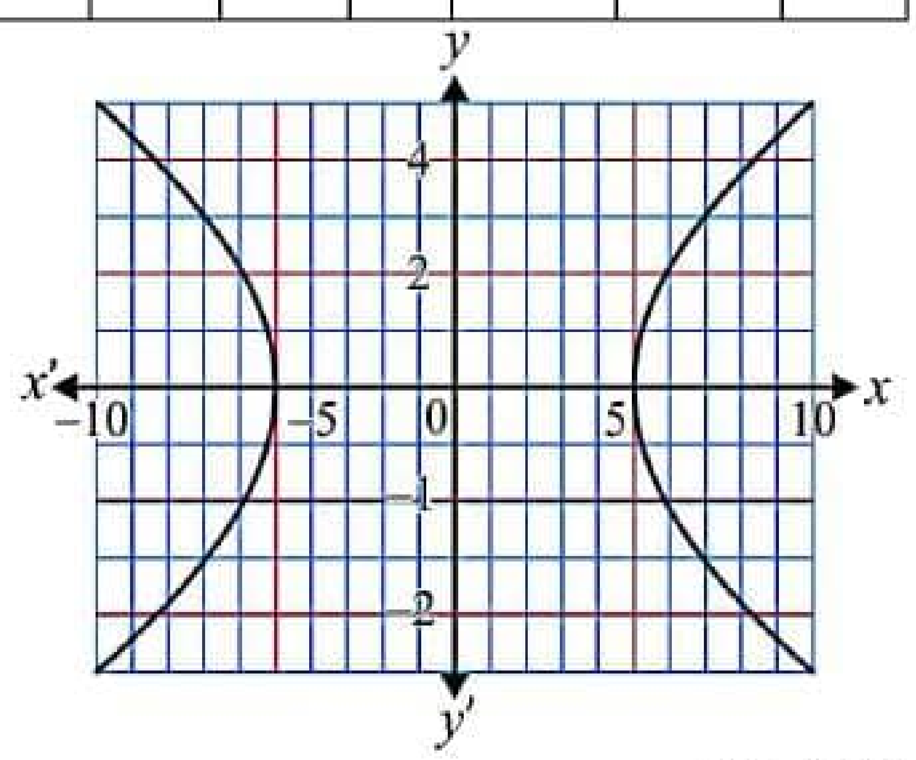


Fig. 2.20

- **Display graphically the discontinuous functions of the type**

$$y = \begin{cases} x & \text{when } 0 \le x < 1 \\ x - 1 & \text{when } 1 \le x \le 2 \end{cases}$$

Here we have a discontinuous function. Let us draw both the function at their respective interval.

- For $y = x$ when $0 \le x < 1$

By constructing a table as $0 \le x \le 1$

x	0	0.2	0.4	0.6	0.8	0.99
$y = x$	0	0.2	0.4	0.6	0.8	0.99

- For $y = x - 1$ when $1 \le x \le 2$

By constructing a table as $1 \le x \le 2$

x	1	1.2	1.4	1.6	1.8	2
$y = x - 1$	0	0.2	0.4	0.6	0.8	1

The graph of discontinuous function as shown in Fig. 2.21.

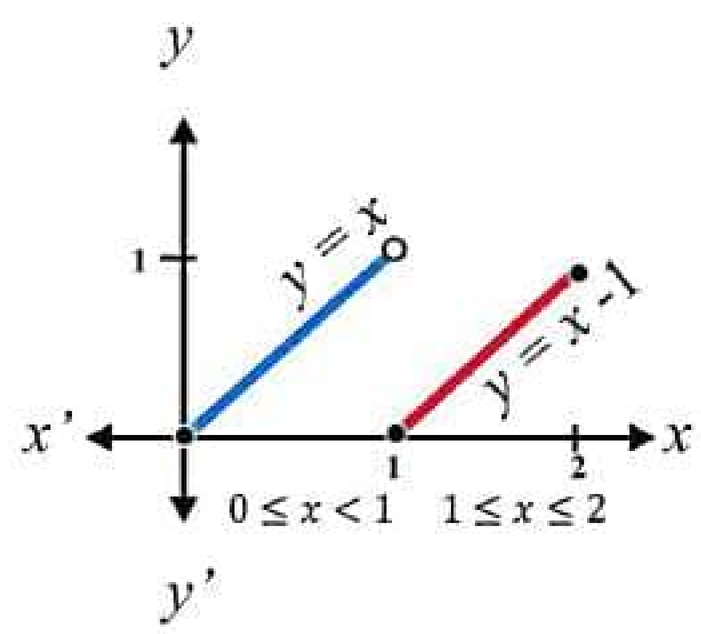


Fig. 2.21

2.5.2 Use MAPLE graphic commands for two-dimensional plot of

- an expression (or a function),
- parameterized form
- implicit function,

  by restricting domain and range of a function

(a) An expression or a function (2D plot using MAPLE)

The standard scale for a Maple plot is x (horizontal axis) ranging from -10 to 10 and the vertical axis is based on the value of the function when x ranges from -10 to 10. The view option allows you to scale the axes in order to see details of interest.

$$> f2 := x \to x^4 + x^3 - 2.x^2 - 3$$

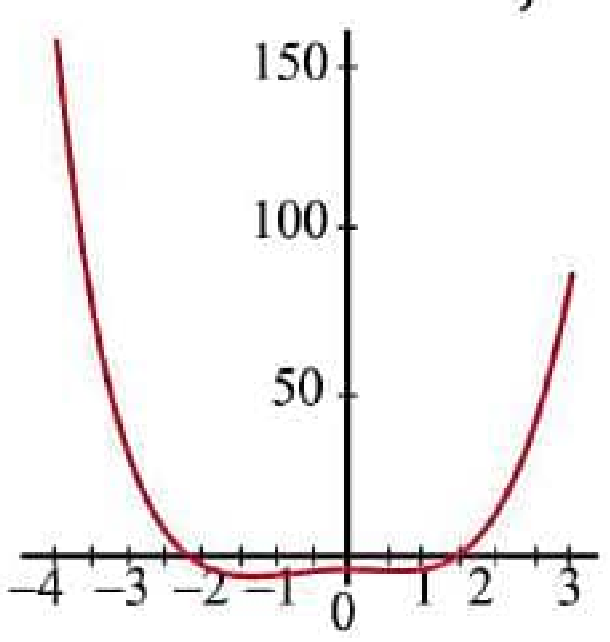


Fig. 2.22

$> plot\ (f2)$

2D plot of the function without x and y ranges

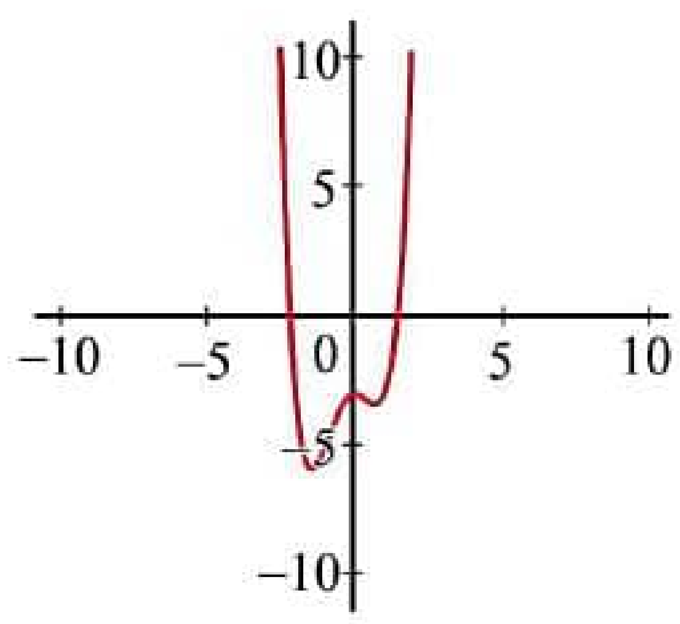


Fig. 2.23

$> plot(f2, view = [-10..10, -10..10])$

2D plot of the function with x and y ranges. (Restricted Domain)

(b) Parameterized form (2D plot using MAPLE)

Maple Command format for 2D plot of Parametric function is as under:

$> plot([x(t), y(t), t = \text{range of t}], h, v, options)$

Where,

$[x, y, range]$ is the parametric specifications

h and v are the horizontal and vertical ranges

Example:

$> plot([t^2, 2t, t = -3..3)])$

$x = t^2, y = 2t$ and range $t = 3..3$

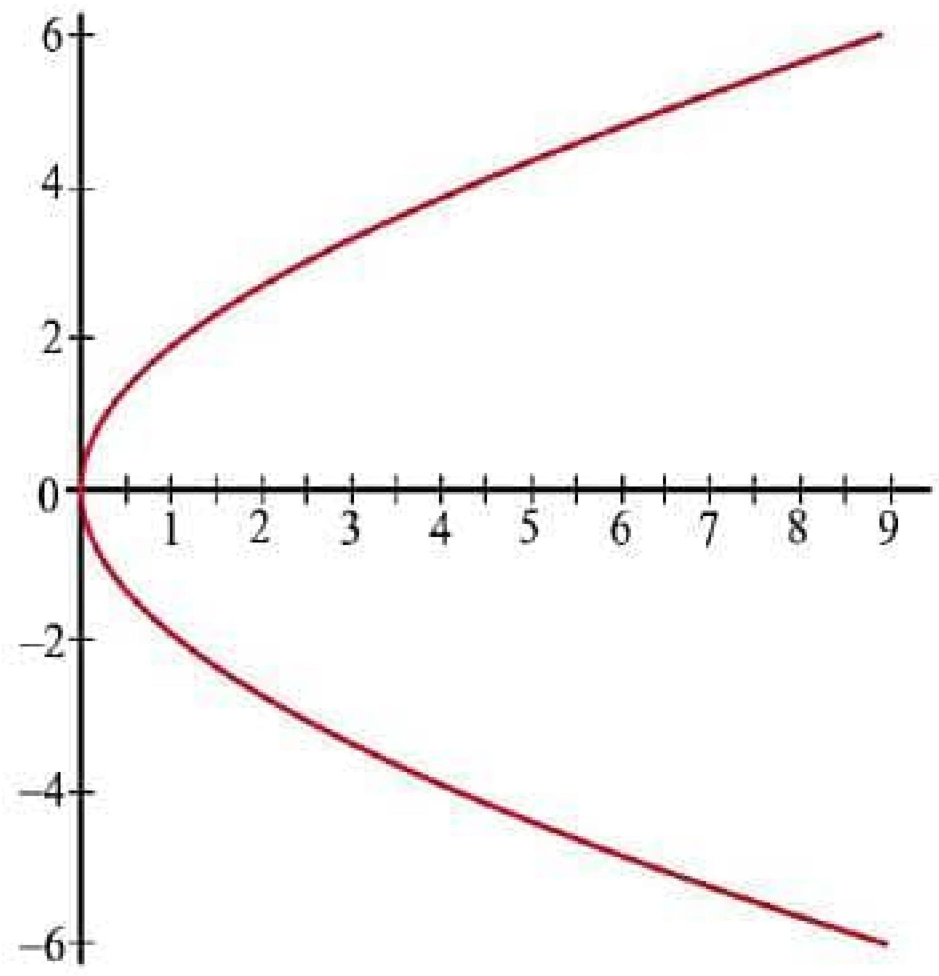


Fig. 2.24

(c) Implicit function form (2D plot using MAPLE)

Maple Command format for 2D plot of implicit function is as under:

$> with(plots, implicitplot)$

$> mplicitplot(f, x = a..b.\ y = c..d, options)$

Where, f is the implicit function

$x = a..b$ and $y = c..d$ are the range on x and y-axis.

Example:

$> with(plots, implicitplot)$

$> impliciitplot$

$\left(\left[\frac{x^2}{16} - \frac{y^2}{9} = 1, x = -10..10, y = -6..6\right)\right]\right)$

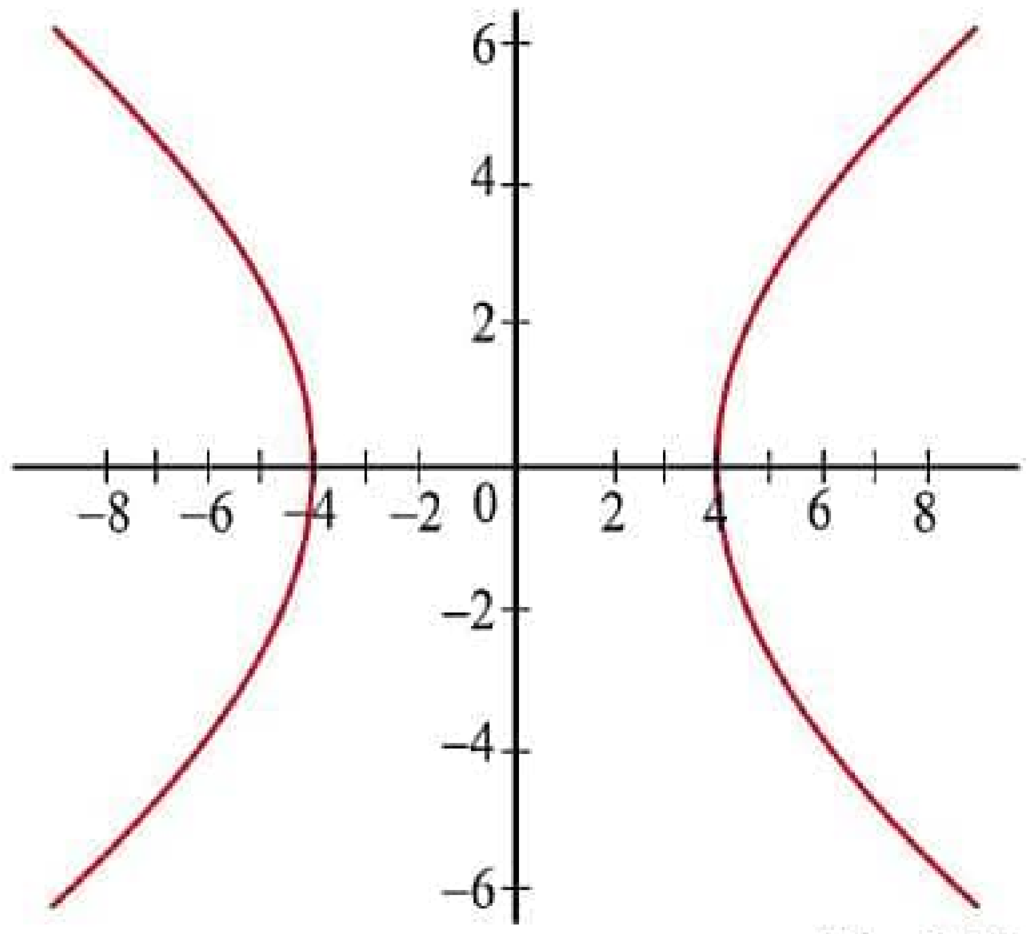


Fig. 2.25

2.5.2 Use MAPLE package plots for plotting different types of functions

Different type of functions is plotted with Maple package. The Maple command format is as under:

$> plot(f, x = x_0..x_1)$

f is a function and $x = x_0..x_1$ is the interval on x-axis.

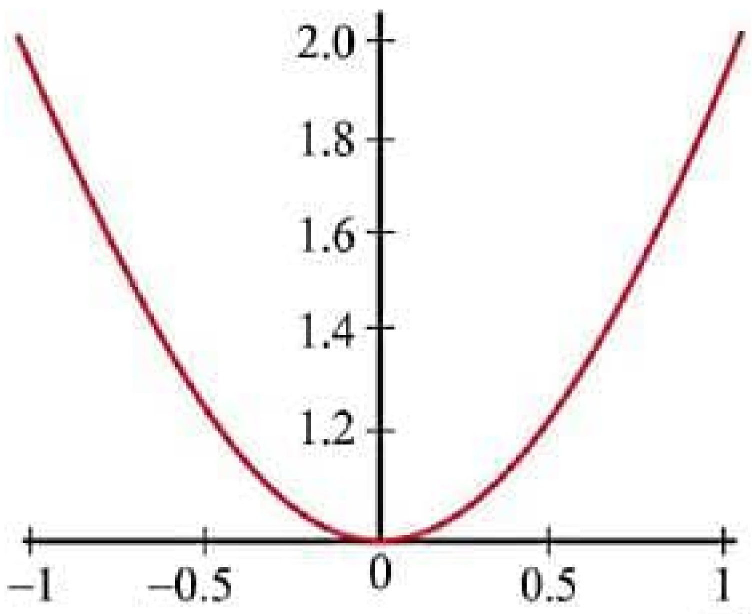


Fig. 2.26

$Plot\ (1 + x^2, x = -1..1)$

Algebraic function

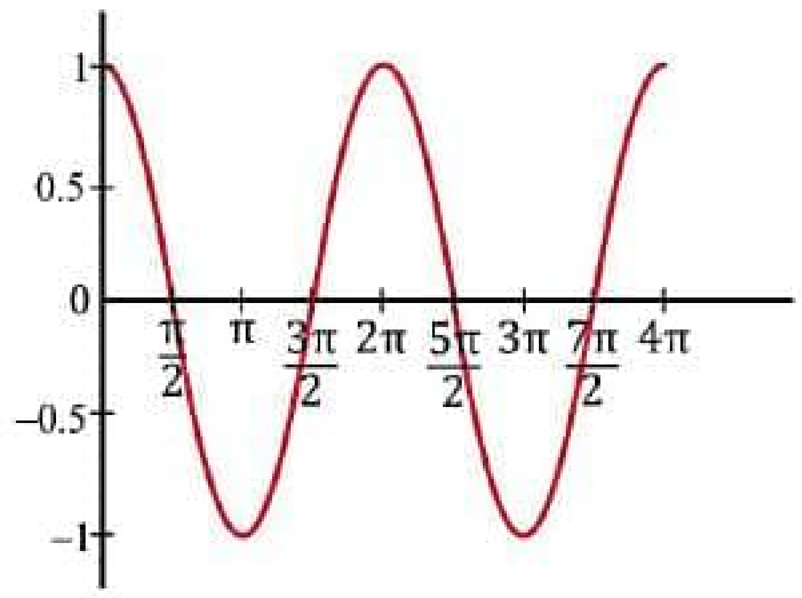


Fig. 2.27

$Plot\ (\cos(x), x = 0..4\pi)$

Trigonometric function

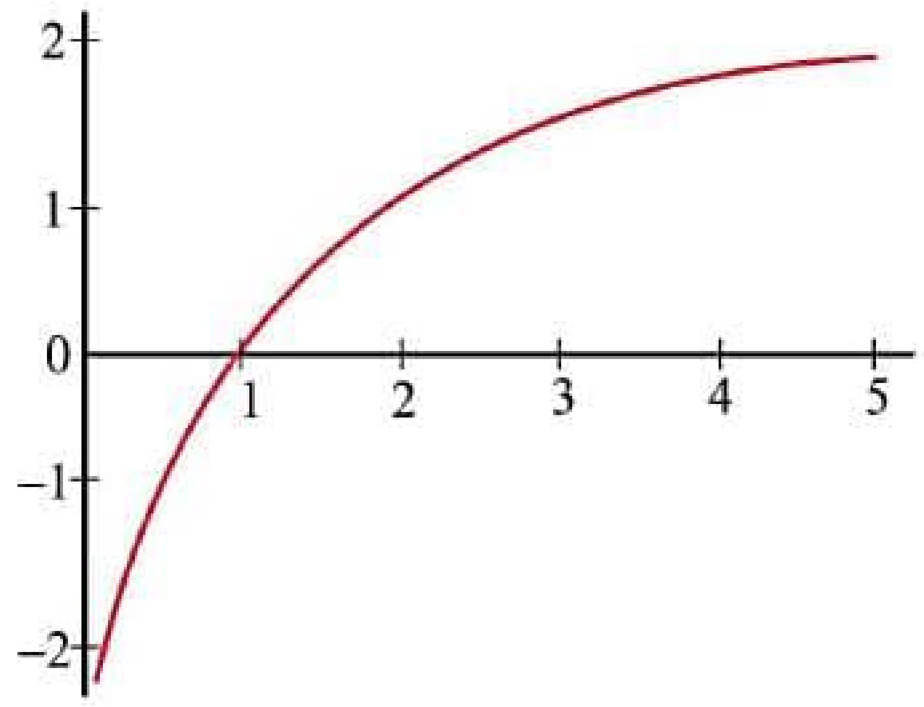


Fig. 2.28

$Plot\ ([\ln(x)], x = -5..5, color = ["Red"])$

Logarithmic function

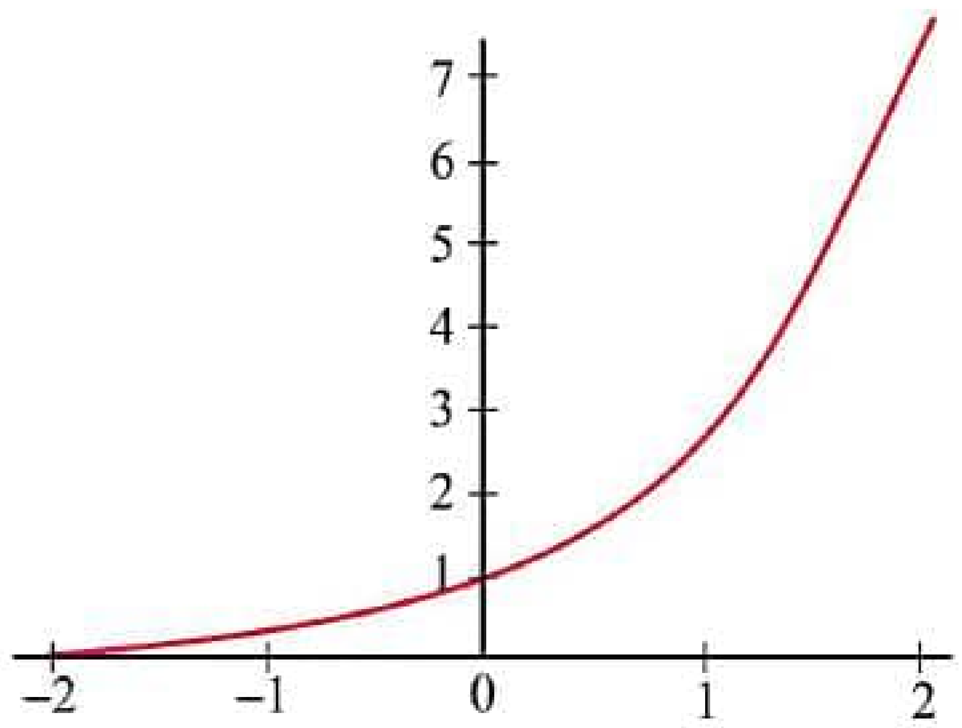


Fig. 2.29

$Plot\ ([\exp(x)], x = -2..2, color = ["Red"])$

Exponential function

Exercise 2.2

1. Which of the following are algebraic, exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions.

(i) $y = x^2 + 5x + 6$ (ii) $f(x) = \tan^{-1}x$

(iii) $y = 2^{x+1}$ (iv) $y = \log_5(x + 2)$

(v) $f(x) = 3\sin x$ (vi) $y = a^{\sin x}$

(vii) $f(x) = \dfrac{x^2 + 5x + 7}{x + 9}$ (viii) $f(x) = \dfrac{\sin x}{\sec x}$

(ix) $y = \log_a \sin x$ (x) $f(x) = \operatorname{cosec}^{-1}\sqrt{x^2 - 1}$

(xi) $f(x) = \tan(\sin x)$ (xii) $y = \dfrac{x}{x + 3}$

(xiii) $f(x) = \sinh x$ (xiv) $y = ln\ cosh\ x$

(xv) $y = \tan h^{-1}x$ (xvi) $y = \cos^{-1}(\ln x)$

2. Identify, whether the y is the explicit or implicit function of independent variable x if:

(i) $xy^2 + 5xy + 7 = 0$ (ii) $y = 3x^2 - 3x + 5$

(iii) $yx^2 + y^2x = 3 - 5y$ (iv) $x^2 + xy^2 = 2 + 3xy$

(v) $y = \dfrac{x+3}{x^2+5}$ (vi) $\dfrac{x}{y} = 3x^3y - 5$

3. Draw the graph of the following functions:

(i) $f(x) = e^{3x}$ (ii) $f(x) = 3log_{10}x$

(iii) $y = \sqrt{36 - x^2}$ (iv) $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

4. Draw the graph of parametric equations of function
$x = at^2,\ y = 2at$, when $a = 4$ and $-5 \le t \le 5$

5. Draw the graph of parametric equations of function
$x = asec\ \theta,\ y = btan\ \theta$, when $a = 3, b = 4$ and $-\pi \le \theta \le \pi$

6. Draw the graph of the $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x & x > 1 \end{cases}$

2.6 Limit of a Function

2.6.1 Identify a real number by a point on the number line

A number which does not involve the square root of negative number is called real number, any real number x can be represented on a straight line by a point P such that the distance of P from a fixed-point O on the line is equal to $|x|$. The straight line is called the number line (Fig. 2.30).

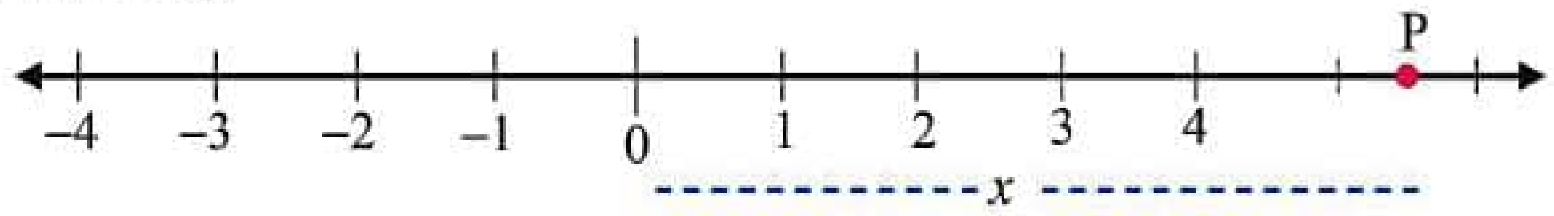


Fig. 2.30

For each real number there is a unique point and conversely for each point of the line, there is a real number, i.e., there is one to one correspondence between the set of real numbers and the set of point on the number line. So, every real number can be identified through a point on the number line.

2.6.2 Define and represent

- **open interval**
- **closed interval**
- **semi open and semi-closed intervals, on the number line**

- **Open interval:**

Let p and q be two real numbers with $p < q$ then the set of all real numbers x such that $p < x < q$ is called an *open interval* and denoted by $]p, q[\ or\ (p, q)$ (i.e., it does not include the endpoints p and q).

i.e., $(p, q) = \{x | x \in \mathbb{R} \wedge p < x < q\}$

and geometrically it is the set of points on the number line between p and q as shown in the Fig. 2.31.

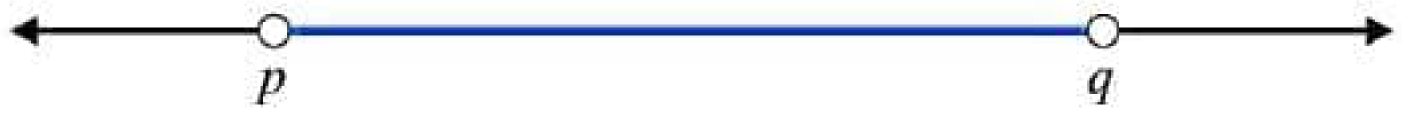


Fig. 2.31

- **Closed interval:**

Closed interval is the set of all real numbers x such that $p \leq x \leq q$ and denoted by $[p, q]$ (i.e, it includes the endpoints p and q),

i.e., $[p, q] = \{x | x \in \mathbb{R} \wedge p \leq x \leq q\}$

and geometrically it is the line segment with end points p and q on the number line as shown in the Fig. 2.32.

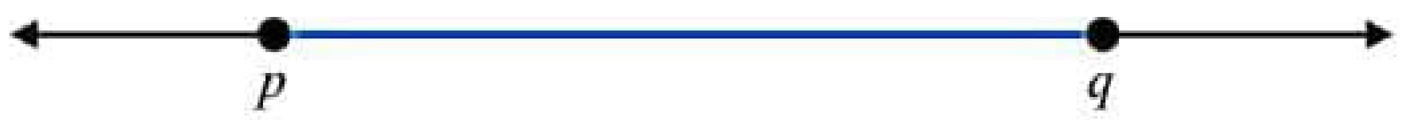


Fig. 2.32

- **Semi open-Semi closed interval**

Semi open- semi closed interval is the set of all real numbers x such that $p < x \leq q$ and is denoted by $(p, q]$. It includes the end point q but not p,

i.e., $(p, q] = \{x | x \in R \wedge p < x \leq q\}$

and geometrically it is the set of all points between p and q where end point q is included and the end point p is excluded on the number line as shown in the Fig. 2.33.

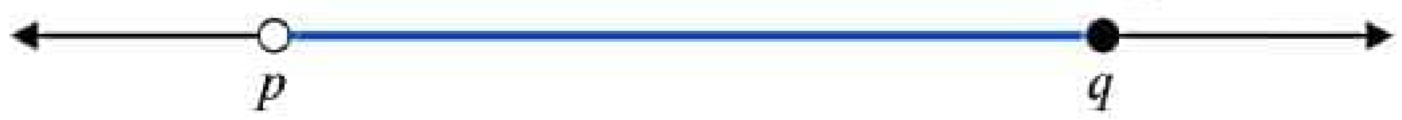


Fig. 2.33

Similarly for $p \leq x < q$, we denote the interval by $[p, q)$ (i.e., it includes the end point p but not q), also defined as: $[p, q) = \{x | x \in R \wedge p \leq x < q\}$

and geometrically it is the set of all points between p and q where end point p is included and the end point q is excluded as shown in the Fig. 2.34.

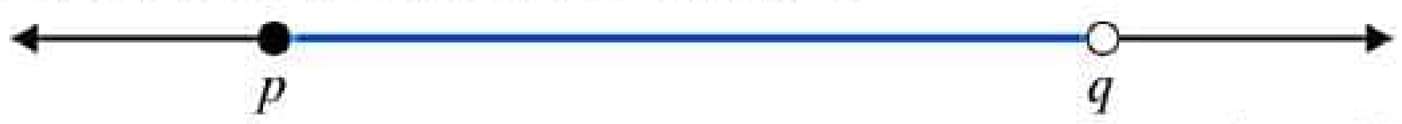


Fig. 2.34

Note:

(i)	$[p, q] - (p, q) = \{p, q\}$	(ii)	$(p, q) - [p, q] = \{\}$
(iii)	$[p, q] \cup (p, q) = [p, q]$	(iv)	$[p, q] \cap (p, q) = (p, q)$
(v)	$\{p, q\} \cup (p, q) = [p, q]$	(vi)	$(-\infty, \infty) = \mathbb{R}$

Examples: Find the following

(i) $[-4, \infty) \cup (-2, 7)$ (ii) $[3, \infty) - (2, \infty)$ (iii) $(2, \infty) \cap (1, 3)$

(iv) $(-\infty,4]-(2,\infty)$, $(-\infty,3)$, $(-4,\infty)$ and $(-\infty,\infty)$

Solution:

(i) $[-4,\infty)\cup(-2,7)=[-4,\infty)$ (ii) $[3,\infty)-(2,\infty)=\{\}$

(iii) $(2,\infty]\cap(1,3)=(2,3)$ (iv) $(-\infty,4]-(2,\infty)=(-\infty,2]$

(v) $(-\infty,4]-(-\infty,3)=[3,4]$ (vi) $(-\infty,4]-(-4,\infty]=(-\infty,-4]$

(vii) $(-\infty,4]-(-\infty,\infty)=\{\}$

2.6.3 Explain the meaning of phrase

- ***x* tends to zero $(x\longrightarrow 0)$,**
- ***x* tends to *a* $(x\longrightarrow a)$**
- ***x* tends to infinity $(x\longrightarrow \infty)$**

Before the definition of the limit of a function, it is necessary to know the clear understanding of the meaning of the following phrases:

- ***x* tends to zero $(x\longrightarrow 0)$**

x tends to zero means x varies in such a way that its numerical value becomes 'closer' to 0 but not 0. Symbolically we write as $x\longrightarrow 0$.

- ***x* tends to *a* $(x\longrightarrow a)$**

x tends to a mean x varies in such a way that the numerical difference of x and a tends to 0. Symbolically $|x-a|\longrightarrow 0 \;\Rightarrow\; x\longrightarrow a$.

- ***x* tends to infinity $(x\longrightarrow \infty)$**

x tends to infinity means x increases without any bound in such a way that no real number exists which is greater than or equal to x. Symbolically, we write $x\longrightarrow\infty$.

2.6.4 Define limit of the sequence

Recall the definition of the sequence, it is a function whose domain is the set of natural numbers. Consider the sequence $a_1, a_2, a_3, \dots a_n, \dots$ denoted by $\{a_n\}$. If the terms of the sequence $\{a_n\}$ getting closer to a specific real number l as n tends to infinity, then l is called the limit of the sequence and is written as

$$\lim_{n\to\infty} a_n = l \text{ or } \lim a_n = l$$

If the value of a_n gets larger and larger without bound as n tends to infinity, then we say limit does not exist and we write

$$\lim_{n\to\infty} a_n = \infty$$

Nevertheless, if value of a_n gets smaller and smaller without bound as n tends to infinity, then limit also does not exist and we write

$$\lim_{n\to\infty} a_n = -\infty$$

For example, the limit of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ will be 0 as each next term of the sequence decreases and becomes closer to 0.

Theorems:

- $\lim(c) = c$ where c is constant
- $\lim(c.a_n) = c.\lim\ a_n$
- $\lim(a_n + b_n) = \lim a_n + \lim b_n$
- $\lim(a_n - b_n) = \lim a_n - \lim\ b_n$
- $\lim(a_n b_n) = \lim a_n \lim\ b_n$
- $\lim\left(\frac{a_n}{b_n}\right) = \frac{lim\ a_n}{lim\ b_n}$

2.6.5 Find the limit of a sequence whose nth term is given

Example 1. Find the limit of the sequence $a_n = \frac{3n^2+5n+7}{5n^2-8n-11}$.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2 + 5n + 7}{5n^2 - 8n - 11}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2\left(3 + \frac{5}{n} + \frac{7}{n^2}\right)}{n^2\left(5 - \frac{8}{n} - \frac{11}{n^2}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 + \frac{5}{n} + \frac{7}{n^2}}{5 - \frac{8}{n} - \frac{11}{n^2}}$$

$$= \frac{\lim_{n\to\infty}(3) + \lim_{n\to\infty}\left(\frac{5}{n}\right) + \lim_{n\to\infty}\left(\frac{7}{n^2}\right)}{\lim_{n\to\infty} 5 - \lim_{n\to\infty}\frac{8}{n} - \lim_{n\to\infty}\frac{11}{n^2}}$$

(Applying limit)

$$= \frac{3 + 0 + 0}{5 - 0 - 0}$$

$$\lim_{n\to\infty} a_n = \frac{3}{5}$$

Example 2. Find the limit of the sequence $a_n = \frac{5n+7}{9n^2+11}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n + 7}{9n^2 + 11}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n\left(5 + \frac{7}{n}\right)}{n^2\left(9 + \frac{11}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{5+\frac{7}{n}}{n\left(9+\frac{11}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{5+0}{n(9+0)} = 0$$

2.6.6 Define limit of a function

Limit of a function $f(x)$ at point a is the number L such that the values of the function get close to L as long as x becomes close enough to the point a.

Mathematically it is written as

$$\lim_{x\to a} f(x) = L$$

For example, to find the limit of the function $f(x) = \frac{1}{4}(x+1)(x-1)(x-5)$ at $x = 3$.

We find all the values of the function when x approaches to 3.

x	2.5	2.55	2.6	2.65	2.7	2.75	2.9	2.95
$f(x)$	-3.28	-3.37	-3.46	-3.54	-3.62	-3.69	-3.89	-3.95

When x approaches to 3, the values of function become close to -4, as shown in the above table.

Hence, the limit of the function at 3 is -4.

i.e., $$\lim_{x\to 3}\left[\frac{1}{4}(x+1)(x-1)(x-5)\right] = -4$$

Note: Let $p(x)$ is polynomial function, then $\lim_{x\to a} p(x) = p(a)$

2.6.7 State the theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples

Let $f(x)$ and $g(x)$ be two functions defined on an open interval containing the number "a". If x approaches "a" both from left and right side of "a", $f(x)$ and $g(x)$ approaches, a specific numbers c and d, called the limit of the function $f(x)$ and $g(x)$ respectively. The same may be written as:

i.e., $\lim_{x\to a} f(x) = c$ and $\lim_{x\to a} g(x) = d$

Following theorems of limits or properties may be applied for finding the limit of the functions:

Theorem 1. (Limit of Sum of Functions)

The limit of the sum of functions is equal to the sum of their limits

i.e., $$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = c + d$$

Theorem 2. (Limit of Difference of Functions)

The limit of the difference of functions is equal to the difference of their limits

i.e., $$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = c - d$$

Theorem 3. (Limit of Product of Functions)

The limit on the Product of functions is equal to the product of their limits

i.e., $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = c.d$

Theorem 4. (Limit of Quotient of Functions)

The limit on the Quotient of functions is equal to the Quotient of their limits

i.e., $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{c}{d}$ where $g(x) \neq 0$

2.7 Important limits

2.7.1 Evaluate the limits of functions of the following types:

- $\frac{x^2-a^2}{x-a}$ and $\frac{x-a}{\sqrt{x}-\sqrt{a}}$ **when** $x \to a$
- $\left(1+\frac{1}{x}\right)^x$ **when** $x \to \infty$
- $(1+x)^{\frac{1}{x}}$, $\frac{\sqrt{x+a}-\sqrt{a}}{x}$, $\frac{a^x-1}{x}$, $\frac{(1+x)^n-1}{x}$ **and** $\frac{\sin x}{x}$ **when** $x \to a$

- $\frac{x^2-a^2}{x-a}, \frac{x-a}{\sqrt{x}-\sqrt{a}}$ **when** $x \to a$

(i) $\lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{x-a}$

$= \lim_{x\to a}(x+a) = 2a$

(ii) $\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

$= \lim_{x\to a}\frac{(x-a)}{\sqrt{x}-\sqrt{a}}\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{x\to a}\frac{(x-a)}{(\sqrt{x})^2-(\sqrt{a})^2}(\sqrt{x}+\sqrt{a})$

$= \lim_{x\to a}\frac{(x-a)}{(x-a)}(\sqrt{x}+\sqrt{a}) = \lim_{x\to a}(\sqrt{x}+\sqrt{a})$

$= 2\sqrt{a}$

- $\left(1+\frac{1}{x}\right)^x$ **when** $x \to \infty$

$$\left(1+\frac{1}{x}\right)^x \quad \text{when} \quad x \to \infty$$

By using Binomial Series

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left[1 + x\left(\frac{1}{x}\right) + \frac{x(x-1)}{2!}\left(\frac{1}{x}\right)^2 + \frac{x(x-1)(x-2)}{3!}\left(\frac{1}{x}\right)^3 + \cdots\right]$$

$$= \lim_{x\to\infty}\left[1+1+\frac{1}{2!}x^2\left(1-\frac{1}{x}\right).\frac{1}{x^2}+\frac{1}{3!}x^3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right).\frac{1}{x^3}+\cdots\right]$$

$$= \lim_{x\to\infty}\left[1-1+\frac{1}{2!}\left(1-\frac{1}{x}\right)+\frac{1}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\cdots\right]$$

when $x\to\infty$, All $\frac{1}{x},\frac{2}{x},\frac{3}{x}$, ... tends to zero.

$$= \left[1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots\right] \qquad \because\ e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$ [Approximate value if e is 2.718281]

- $\boldsymbol{(1+x)^{\frac{1}{x}}}$, $\boldsymbol{\frac{\sqrt{x+a}-\sqrt{a}}{x}}$, $\boldsymbol{\frac{a^x-1}{x}}$, $\boldsymbol{\frac{(1+x)^n-1}{x}}$ **and** $\boldsymbol{\frac{sinx}{x}}$ **when** $\boldsymbol{x\to 0}$

(i) $\boldsymbol{(1+x)^{\frac{1}{x}}}$ **when** $\boldsymbol{x\to 0}$

Let $x = \frac{1}{y}$

As $x\to 0$ then $y\to\infty$

Now, $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$

(ii) $\boldsymbol{\frac{\sqrt{x+a}-\sqrt{a}}{x}}$ **when** $\boldsymbol{x\to 0}$

$$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x} = \lim_{x\to 0}\frac{(\sqrt{x+a}-\sqrt{a})}{x}.\frac{(\sqrt{x+a}+\sqrt{a})}{(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0}\frac{x+a-a}{x}.\frac{1}{(\sqrt{x+a}+\sqrt{a})}$$

$$= \lim_{x\to 0}\frac{1}{(\sqrt{x+a}+\sqrt{a})} = \frac{1}{2\sqrt{a}}$$

(iii) $\boldsymbol{\frac{a^x-1}{x}}$ **when** $\boldsymbol{x\to 0,\ a>0,\ a\neq 1}$

To find $\lim_{x\to 0}\frac{a^x-1}{x}$

Let $a^x-1 = y$ ...(i)

$\Rightarrow$ $a^x = 1+y$

$\Rightarrow$ $x = \log_a(1+y)$

From (i) when $x\to 0$ then $y\to 0$ we have

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)}$$

$$= \lim_{y\to 0}\frac{1}{\frac{1}{y}\ \log_a(1+y)} = \lim_{y\to 0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

$$= \frac{\lim_{y\to 0}(1)}{\lim_{y\to 0}\left[\ \log_a(1+y)^{\frac{1}{y}}\right]}$$

$$= \frac{1}{\log_a \lim_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]} \qquad \because \lim_{x\to a}\log_b\big(f(x)\big) = \log_b\left(\lim_{x\to a} f(x)\right)$$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \frac{1}{\log_a e} = \log_e a = \ln a \qquad \because \left[\lim_{y\to 0}(1+y)^{\frac{1}{y}} = e\right]$$

Corollary: If a is replaced by e, then above formula reduces to

$$\lim_{x\to 0}\frac{e^x-1}{x} = \ln e = 1$$

i.e., $\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$

(iv) $\dfrac{(1+x)^n-1}{x}$ **when** $x \to 0$ **and** $n \in Q$

By using Binomial series

$$(1+x)^n = 1 + n(x) + \frac{n(n-1)}{2!}(x)^2 + \frac{n(n-1)(n-2)}{3!}(x)^3 + \cdots$$

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} = \lim_{x\to 0}\frac{\left[1+n(x)+\frac{n(n-1)}{2!}(x)^2+\frac{n(n-1)(n-2)}{3!}(x)^3+\cdots\right]-1}{x}$$

$$= \lim_{x\to 0}\frac{n(x)+\frac{n(n-1)}{2!}(x)^2+\frac{n(n-1)(n-2)}{3!}(x)^3+\cdots}{x}$$

$$= \lim_{x\to 0}\frac{x\left[n+\frac{n(n-1)}{2!}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots\right]}{x}$$

$$= \lim_{x\to 0}\left[n+\frac{n(n-1)}{2!}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots\right]$$

$$= n$$

(v) $\dfrac{\sin x}{x}$ **when** $x \to 0$

Consider a unit circle with circular sector OAB. If x is the angle measured in radian between the radial segment OA and OB, then it follows from the definitions of the trigonometric functions that $\overline{BD} = \sin x, \overline{OD} = \cos x,$ and $AC = \tan x$,. Also, x is the length of the arc AB from figure 2.35.

We have

Area of ΔODB <Area of sector OAB < Area of ΔOAC

i.e., $\frac{sinx\ cosx}{2} < \frac{x}{2} < \frac{tanx}{2}$

$cosx < \frac{x}{sinx} < \frac{1}{cosx}$ $\left[\text{dividing by } \frac{sinx}{2}\right]$

$\Rightarrow \quad \frac{1}{cosx} > \frac{sinx}{x} > cosx$

or $\quad \cos x < \frac{\sin x}{x} < \frac{1}{cosx}$

$\because \quad \lim_{x\to 0} \frac{1}{\cos x} = 1 = \lim_{x\to 0} \cos x$

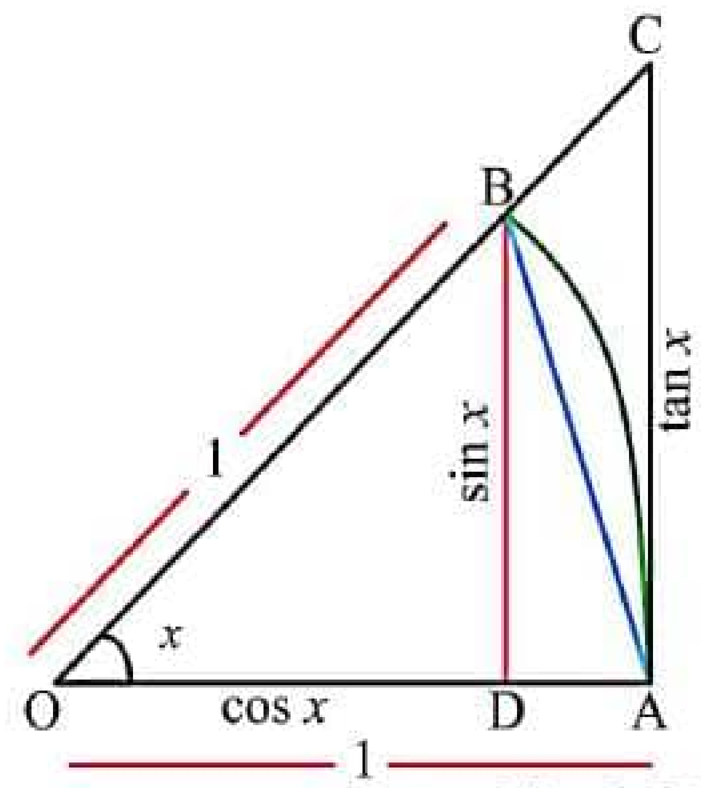


Fig. 2.35

∴ By sandwich theorem, we have

$\therefore \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$

2.7.2 Evaluate limits of different algebraic, exponential and trigonometric functions

(a) The evaluation of the limits of algebraic and exponential functions:

(i) $\lim_{x\to 0} \frac{1}{x-1} = \frac{1}{0-1} = -1$

(ii) $\lim_{x\to \infty} \frac{1}{x+3} = 0$

(iii) $\lim_{x\to 0} (1+\frac{x}{5})^{\frac{3}{x}} = \lim_{x\to 0} \left(1+\frac{x}{5}\right)^{\frac{5}{x}\times\frac{3}{5}}$

$= \left[\lim_{x\to 0} \left(1+\frac{x}{5}\right)^{\frac{5}{x}}\right]^{\frac{3}{5}} = e^{\frac{3}{5}}$ $\quad \because \lim_{x\to 0} [1+x]^{\frac{1}{x}} = e$

(iv) $\lim_{x\to \infty} \left(1+\frac{3}{x}\right)^{\frac{x}{5}} = \lim_{x\to \infty} \left(1+\frac{3}{x}\right)^{\frac{x}{3}\times\frac{3}{5}}$

$= \left[\lim_{x\to \infty} \left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right]^{\frac{3}{5}} = e^{\frac{3}{5}}$ $\quad \because \left[\lim_{x\to \infty} \left[1+\frac{1}{x}\right]^{x} = e\right]$

(v) $\lim_{x\to 0} \frac{3^x-1}{x}$

As we know

$\because \quad \lim_{x\to 0} \frac{a^x-1}{x} = \ln a$

$\therefore \quad \lim_{x\to 0} \frac{3^x-1}{x} = \ln 3$

(vi) $\lim\limits_{x\to 0}\frac{(1+x)^n-1}{5x}$

We know

$\because \quad \lim\limits_{x\to 0}\frac{(1+x)^n-1}{x}=n$

$\therefore \quad \lim\limits_{x\to 0}\frac{(1+x)^n-1}{5x}=\frac{1}{5}\lim\limits_{x\to 0}\frac{(1+x)^n-1}{x}$

$$=\frac{1}{5}(n)=\frac{n}{5}$$

(vii) $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ where $n\in Q$

Let $x=a+h$, as $x\longrightarrow a$, we have $h\longrightarrow 0$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{h\to 0}\frac{(a+h)^n-a^n}{h}=\lim_{h\to 0}\frac{a^n\left(1+\frac{h}{a}\right)^n-a^n}{h}$$

$$=a^n\lim_{h\to 0}\frac{1}{h}\left(1+n\left(\frac{h}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{a}\right)^2+\cdots-1\right)$$

$$=a^n\lim_{h\to 0}\left(\frac{n}{a}+\frac{n(n-1)}{2!}\frac{h}{a^2}+\cdots\right)$$

$$=na^{n-1} \qquad \text{(by applying limit)}$$

(b) The evaluation of the limits of trigonometric Functions:

(i) $\lim\limits_{x\to 0}\frac{\sin 2x}{x}=\lim\limits_{x\to 0}\frac{2\sin 2x}{2x}=2\lim\limits_{2x\to 0}\frac{\sin 2x}{2x}=2$

(ii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\lim\limits_{x\to 0}\frac{1-\cos x}{x}\times\frac{1+\cos x}{1+\cos x}$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{x\sin^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{x}{1+\cos x}.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=0\times 1=0$$

(iii) $\lim\limits_{x\to 0}\frac{\sin 5x}{\sin 3x}=\lim\limits_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}}=\frac{\lim\limits_{x\to 0}\frac{\sin 5x}{x}}{\lim\limits_{x\to 0}\frac{\sin 3x}{x}}=\frac{\lim\limits_{x\to 0}\frac{5\sin 5x}{5x}}{\lim\limits_{x\to 0}\frac{3\sin 3x}{3x}}=\frac{5\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}}{3\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}}=\frac{5}{3}$

(iv) $\lim\limits_{x\to 0}\frac{\pi-x}{\cos(\pi-x)}=\frac{\lim\limits_{x\to 0}(\pi-x)}{\lim\limits_{x\to 0}[\cos(\pi-x)]}=\frac{\pi}{-1}=-\pi$

2.7.3 Use MAPLE command limit to evaluate limit of a function

The format of limit command to evaluate limit of a function in MAPLE are as under:

> limit(f, x = a) for $\left[\lim_{x\to a} f\right]$

> Limit(f, x = a, dir)

Where,

f stands for function whose limit is to be evaluated

X=a stands for $x \to a$

dir means direction i.e., real/complex or left/right of a in $x \to a$

Examples:

$> limit\left(\frac{1}{x}, x = 5\right)$

$\frac{1}{5}$

$> limit\left(\frac{\sin(x)}{x}, x = 0\right)$

1

Directional limits are:

$> limit\left(\frac{1}{x}, x = 3\right)$

$\frac{1}{3}$

$> limit\left(\frac{1}{x}, x = 0, real\right)$

$undefined$

$> limit\left(\frac{1}{x}, x = 0, right\right)$

∞

$> limit\left(\frac{1}{x}, x = 0, left\right)$

$-\infty$

Limit of Piecewise functions:

$> g := piecewise(x < 3, x^2 - 6, 3 \le x, 2x - 1)$

$$g := \begin{cases} x^2 - 6 & x < 3 \\ 2x - 1 & 3 \le x \end{cases}$$

$> limit\ (g, x = 3)$

$undefined$

$> limit\ (g, x = 3, right)$

5

$> limit\ (g, x = 3, left)$

3

Exercise 2.3

1. Find the following:
   (i) $[2,\infty) \cup (3,5)$ (ii) $[-1,1] - (2,\infty)$ (iii) $(5,\infty) \cap (-3,6)$
   (iv) $[3,5] - (3,5)$ (v) $[1,10] \cap [3,11]$ (vi) $(-\infty,5) - (-\infty,3)$
2. Find the nth term and limit of the following sequences
   (i) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (ii) $\frac{1.2}{3.4}, \frac{3.4}{5.6}, \frac{5.6}{7.8}, \ldots$
3. Find the limit of the following sequences whose nth terms are:
   (i) $a_n = \frac{1+5n}{7n}$ (ii) $a_n = \frac{(3n-1)(n^4-n)}{(n^2+5)(n^3-7)}$ (iii) $a_n = \frac{(n+1)!}{n!-(n+1)!}$

4. Find limit of the function $y = \frac{5x}{x+1}$ for $x \to \infty$.

5. Evaluate:

(i) $\lim_{x\to 2} (x^5 + x^2 + x + 1)$ (ii) $\lim_{x\to 5} \left(\frac{1+x}{x^2}\right)$

(iii) $\lim_{x\to 1} [(2x^3 + 3x^2)(x + 1)]$ (iv) $\lim_{x\to 5} \{(x + 1) - (x^2 + 2x + 3)\}$

6. Evaluate the limits of following algebraic and exponential functions:

(i) $\lim_{x\to 1}(x^2 + 4)^3$ (ii) $\lim_{x\to 0} \frac{4}{x-4}$ (iii) $\lim_{x\to\infty} \frac{5}{5x+10}$

(iv) $\lim_{x\to 7} \frac{x^2-49}{x-7}$ (v) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$ (vi) $\lim_{x\to 7} \frac{x-7}{\sqrt{x}-\sqrt{7}}$

(vii) $\lim_{x\to 0} \left(1 + \frac{x}{3}\right)^{\frac{5}{x}}$ (viii) $\lim_{x\to 0}(1 + ax)^{\frac{a}{x}}$ (ix) $\lim_{x\to\infty} \left(1 + \frac{p}{x}\right)^{px}$

(x) $\lim_{x\to\infty} \left(1 - \frac{pq}{x}\right)^{\frac{x}{p}}$ (xi) $\lim_{x\to 0} \frac{17^x-1}{x}$ (xii) $\lim_{h\to 0} \frac{(1+2h)^n-1}{5h}$

(xiii) $\lim_{x\to 0} \frac{2^x-4^x-8^x-1}{x+2}$ (xiv) $\lim_{x\to 0} \left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}}$ (xv) $\lim_{x\to 0} \frac{a^x-b^x}{x}$

(xvi) $\lim_{x\to 0} \frac{e^{-2x}-e^{-11x}}{x}$ (xvii) $\lim_{x\to 0} \frac{3e^{-5x}-5e^{-2x}+2}{x}$

(xviii) $\lim_{x\to 0}(1 + 3\tan x)^{\cot x}$

7. Evaluate the limits of following trigonometric functions:

(i) $\lim_{x\to 0} \frac{a\sin ax}{x}$ (ii) $\lim_{x\to 0} \frac{\sin\sqrt{a}x}{\frac{x}{\sqrt{a}}}$ (iii) $\lim_{x\to 0}(3\cos x + 2\tan x)^3$

(iv) $\lim_{x\to 0} \frac{3\sin x-x^3}{2x}$ (v) $\lim_{x\to 0} \frac{\sin px}{\sin qx}$ (vi) $\lim_{x\to 0} \frac{(2\pi-x)\sec(\pi-x)}{\frac{\pi}{2}}$

(vii) $\lim_{x\to 0} \frac{\sin x^0}{x}$ (viii) $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$ (ix) $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{7x^2}$

8. Evaluate the limits of the following functions:

(i) $\lim_{x\to 0} \frac{\sin^2\left(\frac{x}{2}\right)}{4x^2}$ (ii) $\lim_{x\to\infty} x\sin\frac{1}{x}$ (iii) $\lim_{x\to\infty} \left[\sqrt{x^2 + x + 1} - x\right]$

(iv) $\lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x}$ (v) $\lim_{x\to 0} \frac{a^x+a^{-x}-2}{x^2}$ (vi) $\lim_{x\to 0} \frac{6^x-3^x-2^x+1}{x^2}$

(vii) $\lim_{x\to 3} \frac{\frac{x}{x+2}-\frac{3}{5}}{x-3}$ (viii) $\lim_{y\to 4} \frac{y^{\frac{5}{2}}-16y^{\frac{1}{2}}}{y-4}$ (ix) $\lim_{x\to 1} \frac{\frac{1}{\sqrt{x}}-1}{1-x}$

(x) $\lim_{x\to\pi} \frac{\sqrt{5+\cos x}-2}{\pi-x}$ (xi) $\lim_{x\to 1} x^{\frac{1}{(x-1)}}$ (xii) $\lim_{x\to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(xiii) $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$ (xiv) $\lim_{x\to e} \frac{\ln x-1}{x-e}$

2.8 Continuous and Discontinuous Functions

If the graph of function takes a sudden jump or has a break at $x = x_o$, it is said to be discontinuous function at that point (Fig 2.36), if on the other hand, no such jump occurs then the function is said to be continuous (Fig 2.37).

Discontinuous function

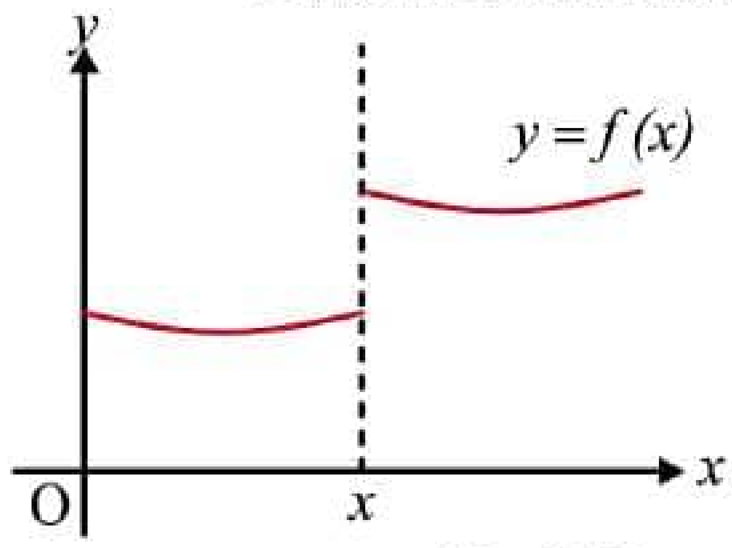


Fig. 2.36

Continuous function

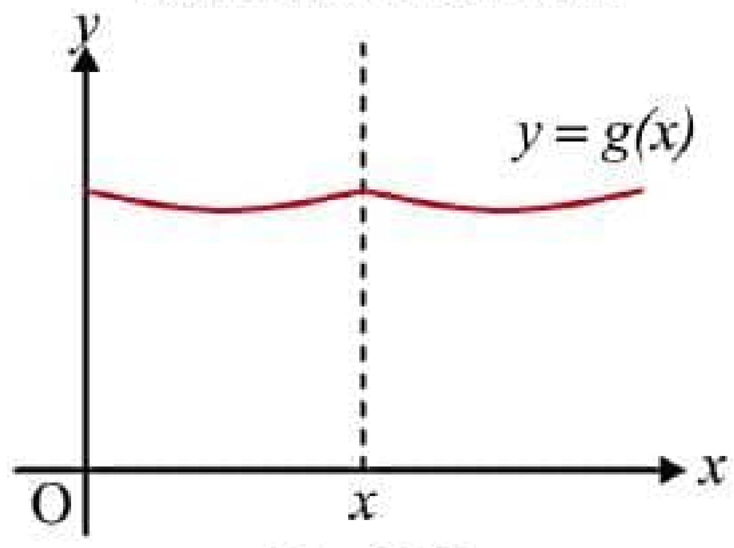


Fig. 2.37

2.8.1 Recognize left hand and right-hand limits and demonstrate through examples

There are two possible limits of the function at any point a. They are left hand limit and right-hand limit. When x approaches "a" from left side, the obtained limit is called left hand limit. It is written as

$$\lim_{x\to a^-} f(x) = m$$

Here m is the left-hand limit of function $f(x)$ at $x = a$.

Similarly, when x approaches "a" from right side, the obtained limit is called right hand limit. It is written as

$$\lim_{x\to a^+} f(x) = n$$

Here n is the right-hand limit of $f(x)$ at $x = a$.

The limit of the function $f(x)$ exists at $x = a$ if both left hand and right-hand limit exist and are equal i.e.,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = m = n = L$$

Example 1. Find left and right-hand limit of $f(x) = \frac{|x|}{x}$ at $x = 0$ and check the existence of the limit.

Solution:

Left hand limit $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} \quad x < 0$

$$= \frac{-x}{x} = -1$$

Right hand limit $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} \quad x > 0$

$$= \frac{x}{x} = 1$$

$\because$ Left hand limit $\neq$ Right hand limit

Limit of the $f(x) = \frac{|x|}{x}$ does not exist at $x = a$.

Example 2. Find left and right-hand limit of $f(x) = \begin{cases} 2x+1, & x < 2 \\ x, & x = 2 \\ 3x-1, & x > 2 \end{cases}$ at $x = 2$ and check the existence of the limit.

Solution:

Left hand limit $= \lim_{x \to 2^-} f(x)$ as $x \longrightarrow 2^- \Rightarrow x < 2$

$\therefore \quad = \lim_{x \to 2^-} (2x + 1)$

$= 2(2) + 1 = 5$

Right hand limit $= \lim_{x \to 2^+} f(x)$ as $x \longrightarrow 2^+ \Rightarrow x > 2$

$\therefore \quad = \lim_{x \to 2^+} (3x - 1)$

$= 3(2) - 1 = 5$

$\because$ Left hand limit = Right hand limit

$\therefore \quad = \lim_{x \to 2} f(x)$ exists.

Example 3. Check the existence of the limit for $f(x) = \frac{x}{\sqrt{1 - \cos x}}$ at $x = 0$.

Solution:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{1 - \cos x} \cdot \sqrt{1 + \cos x}} \quad \left[\text{Multiplying and dividing by } \sqrt{1 + \cos x}\right]$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{(1 - \cos x)(1 + \cos x)}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{1 - \cos^2 x}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{\sqrt{\sin^2 x}}$$

$$= \lim_{x \to 0} \frac{x\sqrt{1 + \cos x}}{|\sin x|}$$

Now, we find the Left hand and Right hand limits to check the existence of the limit.

$$\text{LHL} = \lim_{x \to 0^-} \frac{x\sqrt{1 + \cos x}}{|\sin x|} = \lim_{x \to 0^-} \frac{x\sqrt{1 + \cos x}}{-\sin x} \qquad (\because x < 0 \Rightarrow \sin x < 0)$$

$$= -\lim_{x \to 0^-} \left(\frac{x}{\sin x}\right) \lim_{x \to 0^-} \sqrt{1 + \cos x} \qquad \left(\because \lim_{x \to 0^-} \frac{x}{\sin x} = 1\right)$$

$\therefore \quad = (-1)\sqrt{2} = -\sqrt{2}$

$$\text{RHL} = \lim_{x\to 0^+} \frac{x\sqrt{1+\cos x}}{|\sin x|} = \lim_{x\to 0^+} \frac{x\sqrt{1+\cos x}}{\sin x} \qquad (\because x > 0 \Rightarrow \sin x > 0)$$

$$= \lim_{x\to 0^+} \left(\frac{x}{\sin x}\right) \lim_{x\to 0^+} \sqrt{1+\cos x} \qquad \left(\because \lim_{x\to 0^+} \frac{x}{\sin x} = 1\right)$$

$\therefore \quad = (1)\sqrt{2} = \sqrt{2}$

$\because LHL \neq RHL$

$\therefore \quad \lim_{x\to 0} \frac{x}{\sqrt{1-\cos x}}$ does not exist.

2.8.2 Define continuity of a function at a point and in an interval

(a) Continuity of a function at a point.

A function $f(x)$ is continuous at the point $x = a$ if it satisfies the following conditions:

(i) $f(a)$ is defined i.e, a is in the domain of $f(x)$.

(ii) $\lim_{x\to a} f(x)$ exists.

(iii) $\lim_{x\to a} f(x) = f(a)$

(b) Continuity of a function in an interval:

A function f is continuous over the open interval (a, b) iff it is continuous on every point in (a, b). The function $f(x)$ is continuous over the closed interval $[a, b]$ iff it is continuous on (a, b), the right-hand limit of f at $x = a$ is $f(a)$ and the left-hand limit of f at $x = b$ is $f(b)$.

(c) Discontinuity of a function at point

If a function f is not continuous at a point a then it is said to be discontinuity at a point a. Similarly, if a function is not continuous on interval, then it is called discontinuous on interval.

2.8.3 Test continuity and discontinuity of a function at a point and in an interval.

Example: Test the continuity and discontinuity of the following functions:

(i) $f(x) = \tan x + x^2 + 3x$ at a point $x = 0$

Solution:

Left hand limit $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (\tan x + x^2 + 3x), \qquad x < 0$

$$= (\lim_{x\to 0-} \tan x + \lim_{x\to 0-} x^2 + 3\lim_{x\to 0-} x) \quad \because \tan 0 = 0$$

$$= 0+0+0=0 \qquad \text{[Applying limit]}$$

Right hand limit: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (\tan x + x^2 + 3x), \qquad x > 0$

$$= \left(\lim_{x\to 0+} \tan x + \lim_{x\to 0+} x^2 + 3\lim_{x\to 0+} x\right)$$

$$= 0 + 0 + 0 = 0 \qquad \text{[Applying limit]}$$

Left hand limit $=$ Right hand limit

Limit exists at $x = 0$. Now the value of the function we have $f(x)$ at $x = 0$

$$f(x) = \tan x + x^2 + 3x$$

$$f(0) = \tan 0 + (0)^2 + 3(0)$$

$$f(0) = 0 + 0 + 0 = 0$$

Thus $\lim_{x\to 0} f(x) = f(0)$

So, function is continuous at $x = 0$

(ii) $f(x) = \begin{cases} 2 + x, & \text{when } x < 3 \\ 5 - 2x, & \text{when } x \geq 3 \end{cases}$ at $x = 3$

Solution:

Left hand limit $\lim_{x\to 3-} f(x) = \lim_{x\to 3-} (2 + x), \qquad x < 3$

$$= 2 + 3 = 5 \qquad \text{[Applying limit]}$$

Right hand limit $\lim_{x\to 3+} f(x) = \lim_{x\to 3+} (5 - 2x), \qquad x \geq 3$

$$= 5 - 2(3) = -1 \qquad \text{[Applying limit]}$$

Left hand limit $\neq$ Right hand limit

Limit does not exist at point $x = 3$.

So, the function is discontinuous at $x = 3$

(iii) $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

Solution:

Left hand limit $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}\right), \qquad x < 0$

$$= \lim_{x\to 0^-} \left[\frac{e^{\frac{1}{x}}\left(1 - \frac{1}{e^{\frac{1}{x}}}\right)}{e^{\frac{1}{x}}\left(1 + \frac{1}{e^{\frac{1}{x}}}\right)}\right] = \lim_{x\to 0-} \frac{\left(1 - \frac{1}{e^{\frac{1}{x}}}\right)}{\left(1 + \frac{1}{e^{\frac{1}{x}}}\right)}$$

$$= \frac{(1 - 0)}{(1 + 0)} = 1 \qquad \text{[Applying limit]}$$

Similarly Right-hand limit $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} \left(\dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}\right) = 1, \qquad x > 0$

∵ Left hand limit of $f(x)$ at $x = 0$ is equal to Right hand limit of $f(x)$ at $x = 0$

∵ limit exists at $x = 0$.

Now the value of the function we have $f(x)$ at $x = 0$

$$f(x) = 1 \quad \text{at } x = 0$$

$$f(0) = 1$$

Thus $\lim_{x\to 0} f(x) = f(0)$

So, function is continuous at $x = 0$

Note: Polynomial function are continuous on $(-\infty, \infty)$.

2.8.4 Use MAPLE command iscont to test continuity of a function at a point and in a given interval

In MAPLE we use following commands to test whether the expression or function is continuous or discontinuous at a point and in a given interval.

>iscont (expr, x = a..b)

>iscont (expr, x = a..b, 'closed')

>iscont (expr, x = a..b, 'open')

Where,

expr is an algebraic expression

X is a variable name

a..b is a real interval

'closed' is (optional) indicates that endpoints should be checked

'open' is (optional) indicates that endpoints should not be checked(default)

Examples:

$$> iscont\left(\frac{2}{x+1}, x = 1..2\right)$$
$$true$$

$$> iscont\left(\frac{2}{x+1}, x = -2..1\right)$$
$$false$$

$$> iscont\left(\frac{2}{x+1}, x = -1..1\right)$$
$$true$$

$x = -1..1$ is an open interval, so at point $x = -1$ function is discontinuous but other points of the interval it is continuous. So, it is true.

$$> iscont\left(\frac{1}{x-1}, x = 1..2\right)$$
$$true$$

$$> iscont\left(\frac{1}{x-1}, x = -1..2\right)$$
$$false$$

$$> iscont\left(\frac{1}{x-1}, x = -\infty..\infty\right)$$
$$false$$

$$> iscont\left(\frac{1}{x-1}, x = 0..1\right)$$
$$true$$

$$> iscont\left(\frac{1}{x-1}, x = 0..1\ 'close'\right)$$
$$false$$

$> iscont\ (\sec(x), x = 0..1)$

$true$

$> iscont\ (\sec(x), x = 0..2\pi)$

$false$

$> iscont\ (\text{piecewise}(x < 3, x + 8, 3 \leq x, x^2 + 2), x = 0..\infty)$

$true$

$> iscont\ (\text{piecewise}(x < 3, x + 2, 3 \leq x, x^2 + 2), x = 0..\infty)$

$false$

2.8.5 Application of continuity and discontinuity

We have numerous applications of continuity and discontinuity in our daily life. Few are given below.

(i) If we drop an ice cube in a glass of warm water the temperature of water continuously changes with the time and eventually approaches the room temperature where the glass is stored.

(ii) The human heart is also an example of application of continuity as it beats continuously even the person sleeps.

(iii) The continuous spreading of corona virus however, be controlled or discontinued through precautionary measures such as social distancing, wearing mask and vaccination.

(iv) Population growth is a continuous process and be measured by an exponential function known as population growth model.

Example: The profit obtained by wholesaler of biscuits is given by continuous function $p(x) = \frac{x^2-4}{x-2}$, here x denotes the number of packets of biscuits. Find the profit of wholesaler for selling two packets of biscuits.

Solution: Since, the function $p(x) = \frac{x^2-4}{x-2}$ is continuous, therefore $\lim_{x\to 2} \frac{x^2-4}{x-2} = p(2)$

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2} x + 2 = 4 = p(2)$$

Hence, the wholesaler would obtain profit of Rs. 4 for selling of two packets of biscuits to retailer.

Application of discontinuity of function

Discontinuity of function plays a significant role in various real-life scenarios. Here are some practical applications of discontinuity of function in daily life.

1. Electrical circuits: In electronics and electrical engineering, functions often describe the relationship between voltage, current, and resistance in a circuit. Discontinuities in these functions can represent abrupt changes, such as a switch turning on or off, or a diode transitioning between conducting and non-conducting states.

2. Stock market and finance: Financial data often exhibits discontinuities due to sudden price changes, market openings and closings.

3. Population growth and decay: In demography and biography, functions that describe population growth or decay may experience discontinuities due to sudden events line disease outbreaks, natural disasters, or population control measures.

4. Internet and network traffic: Data transmission rates in computer networks can experience discontinuities when network congestion occurs or when there are abrupt changes in data flow, such as a sudden spike in website traffic.

Exercise 2.4

1. Evaluate the following limits.

   (i) $\lim\limits_{x\to 2^+} \frac{x-2}{|x-2|}$ (ii) $\lim\limits_{x\to 1^-} \frac{x^2+2x-3}{|x-1|}$ (iii) $\lim\limits_{x\to 2} \frac{x^2+4x-12}{|x-2|}$

2. Determine whether $\lim\limits_{x\to 1} f(x)$, $\lim\limits_{x\to 2} f(x)$, $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 4} f(x)$ exit, when

$$f(x) = \begin{cases} 2x+1 & if\ 0 \le x \le 2 \\ x-7 & if\ 2 \le x \le 4 \\ x & if\ 4 \le x \le 6 \end{cases}$$

3. Test the continuity and discontinuity of the following functions.

   (i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at a point $x = 0$

   (ii) $f(x) = \frac{2-\cos 3x - \cos 4x}{x}$ at a point $x = 0$

   (iii) $f(x) = \begin{cases} 7+3x, & \text{when } x < 1 \\ 1-5x, & \text{when } x \ge 1 \end{cases}$ at $x = 1$

4. Determine whether the following function are continuous at $x = 2$

   (i) $f(x) = \frac{x^2-4}{x-2}$ (ii) $g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \ne 2 \\ 3 & \text{when } x = 2 \end{cases}$

   (iii) $h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \ne 2 \\ 4 & \text{when } x = 2 \end{cases}$

5. Suppose that $f(x) = \begin{cases} -x^4 + 3 & \text{when } x \le 2 \\ x^2 + 9 & \text{when } x > 2 \end{cases}$

   Is continuous everywhere justify your conclusion?

6. Find the value of k if $f(x) = \begin{cases} \frac{\sin kx}{x} & , x \ne 0 \\ 2 & , x = 0 \end{cases}$ is continuous at $x = 0$.

7. Find the value of k if $f(x)=\begin{cases} kx-9 & x<5 \\ 9x-k & x>5 \\ 36 & x=5 \end{cases}$ is continuous at $x=5$.

8. Find the values of m and n, so that given function f is continuous at $x=3$.

(i) $f(x)=\begin{cases} mx & if\ x<3 \\ n & if\ x=3 \\ -2x+9 & if\ x>3 \end{cases}$ (ii) $f(x)=\begin{cases} mx & if\ x<3 \\ x^2 & if\ x\geq 3 \end{cases}$

9. If $f(x)=\begin{cases} \frac{\sqrt{2x+5}-\sqrt{x+7}}{2} & , x\neq 2 \\ k & , x=2 \end{cases}$

Find the value of k so that f is continuous at $x=2$.

Review Exercise 2

1. Choose the correct Answer

(i) The function $f(x)=x$ is called -------------

(a) Quadratic function (b) Cubic function
(c) Identity function (d) None of these

(ii) The set $\{x|\ x\in R\}$ can be written as

(a) $[-\infty,\infty]$ (b) $(-\infty,a)$ (c) $(-\infty,\infty)$ (d) (a,∞)

(iii) $[3,5]-(3,5)=$ -------------

(a) $\emptyset$ (b) $(3,5)$ (c) $[3,5]$ (d) $\{3,5\}$

(iv) Let $R=\{(x,y)|\ x,y\in R$ and $y=5\}$

(a) Set of real numbers (b) $\{5\}$ (c) $\emptyset$ (d) $\{1,5\}$

(v) A function $f(x)=\frac{x}{|x|}, x\neq 0$ is an:

(a) odd function (b) even function
(c) linear function (d) none of these

(vi) If $f(x)=4^x$ then $f(x+1)-f(x)=$ -------------

(a) 4 (b) $f(x)$ (c) $3f(x)$ (d) $2f(x)$

(vii) The limit of the sequence having general term $a_n=\frac{1}{2^n}$ is:

(a) 0 (b) 1 (c) ∞ (d) None of these

(viii) $\lim_{x\to\infty}\frac{\sin ax}{x}=?$

(a) $\frac{1}{a}$ (b) a (c) ∞ (d) None of these

(ix) $\lim_{x\to 3}\frac{x^2-9}{x-3}=?$

(a) 3 (b) ∞ (c) 0 (d) 6

(x) If $f:[-1,4]\to\mathbb{R}$ is given by $f(x) = x^2$ then $f(-3)$ is:

(a) 9 (b) -9 (c) does not exit (d) 6

(xi) $\lim_{h\to 0} \sin\left(\frac{\pi}{2}+h\right)$

(a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

(xii) $\lim_{x\to 0} e^{\frac{-1}{x}}$

(a) 0 (b) 1 (c) ∞ (d) $-\infty$

(xiii) $\lim_{x\to c} f(x)$ exits if and only if

(a) $\lim_{x\to c^+} f(x)$ exist (b) $\lim_{x\to c^-} f(x)$ exist

(c) $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$ (d) $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$

(xiv) $\lim_{n\to\infty} \left[1+\frac{1}{n}\right]^{-7n}$

(a) e^{-7} (b) e (c) 1 (d) ∞

(xv) The limit of the sequence $1, \pi^{-1}, \pi^{-2}, \pi^{-3}, \ldots$ is

(a) 1 (b) π (c) ∞ (d) 0

(xvi) Which of the following represents parametric function

(a) $y = f(x)$ (b) $f(x,y) = 0$

(c) $x = f(t), y = g(t)$ (d) None of these

(xvii) If $g(x) = 3x+2$ and $g(f(x)) = x$ then $f(2) =$ ___

(a) 2 (b) 6 (c) 0 (d) 8

(xviii) The value of k for which the function $f(x) = \begin{cases} \frac{x}{\tan 3x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous is

(a) 0 (b) 3 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

(xix) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$

(a) e^3 (b) $e^{\frac{-1}{2}}$ (c) e (d) e^{-1}

(xx) $\text{sech}^{-1} x =$ -----------------

(a) $\ln\left(x+\sqrt{x^2+1}\right)$ (b) $\ln\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right)$

(c) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$

2. If $h(x) = \sqrt{x^2+3}$ and $k(x) = x^2 - 2$, then find composition of function
   (i) $hok(x)$ (ii) koh (iii) hoh (iv) kok
3. If $f(x) = x + 3$ and $g(x) = x^2$, then find $gof(x)$ for $x = 1$.
4. $f(x) = \frac{x+1}{3}$ and $g(x) = 3x + 5$ are two given functions then verify that:
   (i) $(fog)^{-1} = g^{-1}of^{-1}$ (ii) $(gof)^{-1} = f^{-1}og^{-1}$
5. Recognize in the following as explicit or implicit functions and expressed the implicit function as explicit function if possible.
   (i) $x^3 + 2xy = 5x^2 - 3y$ (ii) $3y = 5x^2 - 3x$
   (iii) $x^2y + xy = 5y - 3$ (iv) $2x^2y - xy = 5 + 3xy$
6. Show that the parametric equation $x = a\sec\theta, y = b\tan\theta$. represent the equation of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
7. Draw and explain the graph of the following functions:
   (i) $f(x) = e^{5x}$ (ii) $y = \sqrt{4 - x^2}$ (iii) $\frac{x^2}{9} + \frac{y^2}{16} = 1$
8. Draw the graph of parametric equations of function
   $x = at^2,\ y = 2at$, when $a = 6$ and $-5 \le t \le 5$
9. Draw the graph of parametric equations of function
   $x = a\sec\theta,\ y = b\tan\theta$, when $a = 4, b = 3$ and $-\pi \le \theta \le \pi$
10. Show the following:
    (i) $[1, \infty) \cup (2,3)$ (ii) $[-1,1] \cap (2, \infty)$
    (iii) $(4, \infty) \cap (-3,3)$ (iv) $[2,3] - (2,3)$
    (v) $[1,9] \cap [3,12]$ (vi) $(-\infty, 7) - (-\infty, 2)$
11. Find the limit of the sequence $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$
12. Find the limits of $a_n = -1 + \left(\frac{1}{5}\right)^n$
13. Find limit of the function $y = \frac{1}{x^3}$ for $x \to \infty$
14. Find the value of the following:
    (i) $\lim_{x\to 3} (x^3 + x^4 + x + 5)$ (ii) $\lim_{x\to 7} \left(\frac{1+x}{x^7}\right)$
    (iii) $\lim_{x\to 1} [(2x^4 + 3x^3)(x+3)]$ (iv) $\lim_{x\to 3} [(x+3) - (x^3 + 5x + 5)]$
    (v) $\lim_{x\to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ (vi) $\lim_{x\to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

(vi) $\lim_{x\to 0}(1+2x)^{\frac{5}{x}}$ (viii) $\lim_{x\to 0}\frac{3^{2x}-1}{\sin x}$

15. Determine whether the following functions are continuous at $x=3$.

(i) $f(x)=\frac{x^2-9}{x-3}$ (ii) $g(x)=\begin{cases}\frac{x^2-9}{x-3}, & \text{when } x\neq 3\\ 5 & \text{when } x=3\end{cases}$

(iii) $h(x)=\begin{cases}\frac{x^2-9}{x-3}, & \text{when } x\neq 3\\ 6 & \text{when } x=3\end{cases}$

16. If $f(x)=\begin{cases}3x & if\ x\leq -2\\ x^2-1 & if\ -2<x<2\\ 3 & if\ x\geq 2\end{cases}$. Discuss the continuity or discontinuity at $x=2$ and $x=-2$.

(i) $f(x)=\frac{x^2-9}{x-3}$ (ii) $g(x)=\begin{cases}\frac{x^2-9}{x-3}, & \text{when } x\neq 3\\ 5 & \text{when } x=3\end{cases}$

(iii) $h(x)=\begin{cases}\frac{x^2-9}{x-3}, & \text{when } x\neq 3\\ 6 & \text{when } x=3\end{cases}$

Differentiation

Unit 3

Introduction

The differential calculus is the branch of mathematics developed by Isaac Newton and Gottfried Wilhelm Leibniz (G. W. Leibniz). This branch is concerned with the problems of finding the rate of change of function with respect to the variable on which it depends.

3.1 Derivative of a Function

3.1.1 Distinguish between independent and dependent variables

An independent variable is a variable whose value never depends on another variable, whereas a dependent variable is a variable whose values depends on another variable.

The equation $y = f(x)$ is a general notation which expresses the relation between the two variables x and y, where y depends on x.

e.g., in function $y = f(x) = 3x + 4$, x is the **independent variable** and y is **dependent variable**.

3.1.2 Estimate corresponding change in the dependent variable when independent variable is incremented (or decremented)

Let $y = f(x)$ is a function with dependent variable y and independent variable x. If Δx is the small change in the independent variable x then corresponding change in y will be Δy

i.e., $\Delta y = f(x + \Delta x) - f(x)$

Similarly, when independent variable x is decremented then corresponding change in y will be Δy

i.e., $\Delta y = f(x) - f(x - \Delta x)$

Example 1. $y = x^3 + 1$ then calculate the corresponding change in y when x is incremented from 1 to 1.01.

Solution: Since x is incremented from 1 to 1.01, therefore the change in independent variable x is $\Delta x = 1.01 - 1$

$\Delta x = 0.01$

Now, the corresponding change in y will be

$$\Delta y = f(x + \Delta x) - f(x)$$
$$\Delta y = f(1 + 0.01) - f(1)$$
$$\Delta y = f(1.01) - f(1)$$

$$= ((1.01)^3 + 1) - ((1)^3 + 1)$$
$$= 2.03031 - 2$$
$$\Delta y = 0.030301$$

Thus, the corresponding change in y when x is incremented from 1 to 1.01 is 0.030301.

Example 2. If $y = e^x$ calculate the corresponding change in y, when x is decremented from 2 to 1.98.

Solution: Since, x is decremented from 2 to 1.98, therefore

$$\Delta x = 2 - 1.98$$
$$\Delta x = 0.02$$

Now, the corresponding change in y will be

$$\Delta y = f(x) - f(x - \Delta x)$$
$$\Delta y = f(2) - f(2 - 0.02)$$
$$\Delta y = f(2) - f(1.98)$$
$$\Delta y = e^2 - e^{1.98}$$
$$\Delta y \approx 7.38905 - 7.24274$$
$$\Delta y \approx 0.14631.$$

Thus, the corresponding change in y when x is decrements from 2 to 1.98 is 0.14631.

3.1.3 Explain the concept of rate of change

The rate of change is the speed at which a dependent variable changes with respect to an independent variable. It can generally be expressed as a ratio of change in dependent variable and change in independent variable. Let $y = f(x)$ is function and Δx and Δy are the changes in independent variable x and dependent variable y respectievly. Now,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the rate of change of y with respect to x, which is commonly known as average rate of change. However, when Δx is very small that is $\Delta x \longrightarrow 0$, then such rate of change is called instantaneous rate of change.

Example 1. If $y = x^2 - 6x + 8$ determine the average rate of change of y which respect to x when x varies from 1 to 1.3.

Solution: Given function is

$$y = f(x) = x^2 - 6x + 8$$

and $\Delta x = 1.3 - 1 = 0.3$

Now, average rate of change of y with respect of x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(1 + 0.3) - f(1)}{0.3}$$

$$\frac{\Delta y}{\Delta x}=\frac{f(1.3)-f(1)}{0.3}$$
$$\frac{\Delta y}{\Delta x}=\frac{((1.3)^2-6(1.3)+8)-((1)^2-6(1)+8)}{0.3}$$
$$\frac{\Delta y}{\Delta x}=\frac{1.89-3}{0.3}$$
$$\frac{\Delta y}{\Delta x}=\frac{-1.11}{3}=-3.7$$

Thus, the required rate of change is -3.7.

Exercise 3.1

1. Find the average rate of change of the following functions when x varies from a to b.
   (i) $y=f(x)=x^2+4$; $a=2, b=2.3$
   (ii) $y=f(x)=x^3-4$; $a=2, b=2.3$
   (iii) $y=f(x)=x^3-8$; $a=3, b=2.5$
2. Find out the average rate of change when x changes from a to b.
   (i) $A=\pi x^2$, where x is the radius of the sphere; $a=3, b=3.1$
   (ii) $V=\frac{4}{3}\pi x^3$, where x is the radius of the circle; $a=2, b=1.9$
3. The price p in rupees after "t" years is given by $p(t)=3t^2+t+1$. Find the average rate of change of inflation from $t=3$ to $t=3.5$ years.
4. A ball is thrown vertically up, its height in metres after t seconds is given by the formula $h(t)=-16t^2+80t$. Find the average velocity when t changes from a to b.
   (a) $a=2, b=2.1$ (b) $a=2, b=2.01$

3.1.4 Define derivative of a function as an instantaneous rate of change of a variable with respect to another variable

The instantaneous rate of change of dependent variable y with respect to x is called the derivative of the function $y=f(x)$.

For example, consider displacement of an object is the function of time i.e., $s=f(t)$. Now, instantaneous rate of displacement with respect to time is called velocity and it is the derivative of displacement with respect to time.

3.1.5 Define derivative or differential coefficient of a function

Let $y=f(x)$ the derivative of $f(x)$ is the limit of ratio of increments δy and δx at zero i.e., $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}$ and it is denoted by $f'(x)\frac{dy}{dx},\frac{d}{dx}f(x)$ or y'.

A real valued function $f(x)$ is said to be derivable or differentiable at x, iff $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}$ exists where δy and δx are the increments in y and x respectively.

i.e., $f(x)=\lim\limits_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ exists ...(i)

The derivative of a function $f(x)$ at any point a is denoted by $f'(a)$ is defined as:

$$f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(x)}{(a+\delta x) - a},$$

Now, if we substitute, $x = a + \delta x$ and $x = a$, with new limits $x \to a$ as $\delta x \to 0$, then;

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \ldots\text{(ii)}$$

Note: The process of finding derivative is called the differentiation and to find the derivative by either (i) or by (ii) is called ab-initio method/ first principle or by method of definition.

Note: $\frac{dy}{dx}$ does not mean the ratio of dy and dx i.e., $\frac{dy}{dx} \neq dy \div dx$

$\frac{dy}{dx}$ means derivative of y w.r.t. x, i.e., $\frac{d}{dx}(y)$, $\frac{d}{dx}$ is a differential operator.

Example 1. Find derivative of $y = x^2 + 2$ w.r.t x by definition

Solution: Given that

$y = f(x) = x^2 + 2$

$\therefore f(x+\delta x) = (x+\delta x)^2 + 2$

By definition, we mean that:

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - f(x)}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{x^2 + 2x.\delta x + (\delta x)^2 + 2 - x^2 - 2}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta x\,(2x + \delta x)}{\delta x}$$

$$\Rightarrow f'(x) = \frac{dy}{dx} = 2x + (0) = 2x$$

Thus, derivative of $x^2 + 2$ is $2x$.

Example 2. Find the derivative of $\sqrt{x}$ by definition.

Solution: Given that

$y = f(x) = \sqrt{x}$

$\therefore\ y + \delta y = f(x+\delta x) = \sqrt{x+\delta x}$

By definition

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore\ f'(x)=\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}$$

$$\Rightarrow f'(x)=\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\delta x\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\Rightarrow f'(x)=\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{x+\delta x-x}{\delta x\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\Rightarrow f'(x)=\frac{dy}{dx}=\frac{1}{\left(\sqrt{x+0}+\sqrt{x}\right)}=\frac{1}{2\sqrt{x}}$$

Thus, derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$

3.1.6 Differentiate $y=x^n$, where $n\in Z$ (the set of integers), from first principle (the derivation of power rule)

Case-I: Let $y=f(x)=x^n$, where n is positive integer

$$\therefore\ y+\delta y=f(x+\delta x)=(x+\delta x)^n\ ,$$

By definition of derivative,

$$\because\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{(x+\delta x)^n-x^n}{\delta x}$$

$$\Rightarrow\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{x^n+\binom{n}{1}x^{n-1}(\delta x)+\binom{n}{2}x^{n-2}(\delta x)^2+\cdots+(\delta x)^n-x^n}{\delta x}$$

(apply binomial theorem)

$$\Rightarrow\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{\delta x}{\delta x}\left[nx^{n-1}+\cdots+(\delta x)^{n-1}\right]$$

$$\Rightarrow\ \frac{dy}{dx}=f'(x)=nx^{n-1}+0+0+\cdots+0=nx^{n-1}$$

Thus, $\boxed{f'(x)=\frac{d}{dx}(x^n)=nx^{n-1}}$

Case-II: Let $y=f(x)=x^n$ when n is negative integer

$$\therefore y+\delta y=f(x+\delta x)=(x+\delta x)^n$$

By definition of derivative,

$$\because\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\therefore\ \frac{dy}{dx}=f'(x)=\lim_{\delta x\to 0}\frac{(x+\delta x)^n-x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{x^n\left(1+\frac{\delta x}{x}\right)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \frac{\left(1+\frac{\delta x}{x}\right)^n - 1}{\delta x}$$

using binomial series, we have,

$$\frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \frac{\left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right] - 1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^n . \lim_{\delta x \to 0} \left(\frac{\delta x}{x} . \frac{1}{\delta x}\right)\left[n + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{n-1} . \lim_{\delta x \to 0} \left[n + \frac{n\,(n-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{n-1} . \lim_{\delta x \to 0} [n + 0 + 0 + \cdots] = nx^{n-1}$$

Thus, $f'(x) = \frac{d}{dx}(x^n) = nx^{n-1} \quad \forall\, n \in Z$

3.1.7 Differentiate $y = (ax + b)^n$, where $n = \frac{p}{q} \in Q$ and p & q are integers such that $q \neq 0$, from first principle.

Let $\quad y = f(x) = (ax + b)^n$, where $n = \frac{p}{q} \in Q$ and $q \neq 0$.

$\therefore y + \delta y = f(x + \delta x) = [a(x + \delta x) + b]^n$,

By definition of derivative,

$$\because \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[a(x + \delta x) + b]^n - (ax + b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[(ax + b) + a\delta x]^n - (ax + b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(ax + b)^n \left[1 + \frac{a\delta x}{(ax + b)}\right]^n - (ax + b)^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax + b)^n . \lim_{\delta x \to 0} \frac{[1 + \frac{a\delta x}{(ax + b)}]^n - 1}{\delta x}$$

Using binomial series, we have

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax + b)^n . \lim_{\delta x \to 0} \frac{[1 + n\,\frac{a\delta x}{ax + b} + \frac{n(n-1)}{2!} . \left(\frac{a\delta x}{ax + b}\right)^2 + \ldots - 1]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^n . \lim_{\delta x \to 0} \frac{a\delta x}{(ax+b)} \cdot \frac{1}{\delta x}\left[n + \frac{n(n-1)}{2!} \cdot \frac{a\delta x}{(ax+b)^2} + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^{n-1} . a \lim_{\delta x \to 0}\left[n + \frac{n(n-1)}{2!} \cdot \frac{a\delta x}{(ax+b)} + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (ax+b)^{n-1} . a \lim_{\delta x \to 0} [n + 0 + 0 + \cdots]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = na(ax+b)^{n-1}$$

Thus, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$.

Examples: Find derivative of the following the w.r.t. x by first principle.

(a) $2x^5 + 1$ (b) x^{-3} (c) $(2x+5)^{\frac{5}{2}}$

Solutions (a): Let $y = f(x) = 2x^5 + 1$

$\therefore y + \delta y = f(x + \delta x) = 2(x + \delta x)^5 + 1$

definition of derivative, we have

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{[2(x+\delta x)^5 + 1] - (2x^5 + 1)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{2(x+\delta x)^5 - 2x^5}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \lim_{\delta x \to 0} \frac{(x+\delta x)^5 - x^5}{\delta x}$$

Using binomial theorem, we have,

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \lim_{\delta x \to 0}\left[x^5 + 5x^4(\delta x) + \frac{5.4}{2!} x^3(\delta x)^2 + \cdots + (\delta x)^5 - x^5\right] . \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2. \frac{(5.\delta x)}{\delta x} \lim_{\delta x \to 0}\left[x^4 + \frac{4}{2!} x^3 . \delta x + \cdots + (\delta x)^4\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10. \lim_{\delta x \to 0}\left[x^4 + \frac{4}{2!} x^3 . \delta x + \cdots + (\delta x)^4\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10(x^4 + 0 + 0 + \cdots +)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 10x^4$$

Thus, $\frac{d}{dx}(2x^5 + 1) = 10x^4$

Solution (b): Let $y = f(x) = x^{-3}$

$$\therefore y + \delta y = f(x + \delta x) = (x + \delta x)^{-3}$$

By definition of derivative, we have,

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{(x + \delta x)^{-3} - x^{-3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{x^{-3}\left(1 + \frac{\delta x}{x}\right)^{-3} - x^{-3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{-3}. \lim_{\delta x \to 0} \frac{\left(1 + \frac{\delta x}{x}\right)^{-3} - 1}{\delta x}$$

Using binomial series, we have

$$\therefore \frac{dy}{dx} = f'(x) = x^{-3}. \lim_{\delta x \to 0} \left[\left\{1 + (-3)\left(\frac{\delta x}{x}\right) + \frac{(-3)(-4)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right\} - 1\right].\frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = x^{-3}.\left(\frac{-3\delta x}{x}.\frac{1}{\delta x}\right). \lim_{\delta x \to 0} \left[1 + \frac{(-4)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -3x^{-4}$$

Thus, $\boxed{\frac{d}{dx}\left(x^{-3}\right) = -3x^{-4}}$

Solution (c): Given that

$$f(x) = y = (2x + 5)^{\frac{5}{2}}$$

$$f(x + \delta x) = (2x + 5 + 2\delta x)^{\frac{5}{2}}$$

Now, by using the definition of derivatives

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + 2\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(2x + 5 + 2\delta x)^{\frac{5}{2}} - (2x + 5)^{\frac{5}{2}}}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(2x + 5)^{\frac{5}{2}}\left[\left(1 + \frac{2\delta x}{2x + 5}\right)^{\frac{5}{2}} - 1\right]}{\delta x}$$

$$= (2x + 5)^{\frac{5}{2}} \lim_{\delta x \to 0} \frac{1}{\delta x}\left[1 + \frac{5}{2}\left(\frac{2\delta x}{2x + 5}\right) + \frac{5}{2}\cdot\frac{3}{2}\left(\frac{2\delta x}{2x + 5}\right)^2 + \cdots - 1\right]$$

$$= (2x+5)^{\frac{5}{2}} \lim_{\delta x \to 0} \frac{\delta x}{\delta x}\left(\frac{5(2)}{2(2x+5)} + \frac{5}{2}\cdot\frac{3}{2}\left(\frac{\delta x}{2x+5}\right)^{1} + \cdots\right)$$

$$= (2x+5)^{\frac{5}{2}} \cdot 5 \frac{1}{(2x+5)} = 5(2x+5)^{\frac{3}{2}}$$

Exercise 3.2

1. Find by definition (ab-initio) the derivatives w.r.t "x" of the following functions defined as:

(i) $f(x) = 2x$ (ii) $f(x) = 1 - \sqrt{x}$ (iii) $\frac{1}{\sqrt{x}}$

(iv) $f(x) = 3 - x^2$ (v) $f(x) = x(x+1)$ (vi) $f(x) = x^2 - 3$

(vii) $f(x) = x^3 + 5$ (viii) $f(x) = 4x^2 - 3x$

(ix) $f(x) = \frac{1}{x+2}$ (x) $f(x) = \frac{3}{2x+5}$

2. Find $f'(x)$ for the following functions using definition:

(i) $f(x) = \sqrt[3]{2x+1}$ (ii) $f(x) = (2x-1)^{-\frac{1}{2}}$

(iii) $f(x) = (6x+7)^{\frac{5}{2}}$ (iv) $f(x) = (3x-5)^{-\frac{3}{2}}$

3.2 Theorems on differentiation

We will prove different theorems for differentiation.

- **The derivative of a constant is zero.**

Proof: Let $y = f(x) = c$ ... (i), where c is constant

$\therefore\ y + \delta y = f(x + \delta x) = c$

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{c - c}{\delta x} = 0$$

Thus, $\frac{d}{dx}(c) = 0$

Hence proved.

- **The derivative of any constant multiple of a function is equal to the product of that constant and derivative of the function.**

i.e., $\frac{d}{dx}[a\,f(x)] = a\frac{d}{dx} f(x) = af'(x)$.

Proof: Let $y = af(x) = g(x)$, (say)

$\therefore\ y + \delta y = af(x + \delta x) = g(x + \delta x)$

$$\because \frac{dy}{dx} = g'(x) = \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = \lim_{\delta x \to 0} \frac{a f(x+\delta x) - a f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = a \cdot \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x},$$

$$\Rightarrow \frac{dy}{dx} = g'(x) = af'(x), \qquad \left[\because f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

Thus, $\boxed{\frac{d}{dx}[af(x)] = af'(x)}$

Hence proved.

- **The derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.**

i.e., $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$

Proof: Let $y = h(x) = f(x) \pm g(x)$

$\therefore y + \delta y = h(x + \delta x) = f(x+\delta x) \pm g(x+\delta x)$

$$\because \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) \pm g(x+\delta x) \pm [f(x) \pm g(x)]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] \pm \lim_{\delta x \to 0} \left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$\frac{dy}{dx} = h'(x) = f'(x) \pm g'(x)$$

Thus, $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$

Hence proved.

- **The derivative of a product of two functions is equal to (The first function) × (derivative of second function) plus (derivative of the first function) × (the second function).**

i.e., $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$

$= f(x).g'(x) + g(x).f'(x)$

Proof: Let $y = h(x) = f(x) \cdot g(x)$, (say)

$\therefore y + \delta y = h(x + \delta x) = f(x+\delta x).g(x+\delta x)$

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x).g(x+\delta x) - f(x).g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x).g(x+\delta x) - f(x+\delta x).g(x) + f(x+\delta x).g(x) - f(x).g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} f(x+\delta x) . \lim_{\delta x \to 0} \left[\frac{g(x+\delta x) - g(x)}{\delta x}\right] + \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] . \lim_{\delta x \to 0} g(x)$$

$$\Rightarrow \frac{dy}{dx} = f(x+0) . g'(x) + f'(x) . g(x)$$

$$\Rightarrow \frac{dy}{dx} = f(x). g'(x) + g(x) . f'(x)$$

Thus, $\frac{d}{dx}[f(x) . g(x)] = f(x). g'(x) + f'(x). g(x)$

This is known as product rule for differentiation of two functions.

Hence Proved.

- **The derivative of a quotient of two functions is equal to denominator times the derivative of the numerator minus the numerator times the derivative of the denominator and all divided by the square of the denominator.**

i.e., $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) . \frac{d}{dx} f(x) - f(x) . \frac{d}{dx} g(x)}{[g(x)]^2} = \frac{g(x) . f'(x) - f(x) . g'(x)}{[g(x)]^2}$$

Proof: Let $y = h(x) = \frac{f(x)}{g(x)}$ (say)

Let $y = h(x) =$

$\therefore \quad y + \delta y = h(x + \delta x) = \frac{f(x+\delta x)}{g(x+\delta x)}$

$\because \quad \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \frac{h(x+\delta x) - h(x)}{\delta x}$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)}\right] . \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{g(x) . f(x+\delta x) - g(x+\delta x) . f(x)}{\delta x . g(x+\delta x) . g(x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \lim_{\delta x \to 0} \left[\frac{g(x) . f(x+\delta x) - f(x).g(x) + f(x).g(x) - g(x+\delta x) . f(x)}{\delta x . g(x+\delta x) . g(x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x) \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] - f(x) . \lim_{\delta x \to 0} \left[\frac{g(x+\delta x) - g(x))}{\delta x}\right]}{\lim_{\delta x \to 0} . g(x+\delta x) . g(x)}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x)\,.\frac{d}{dx}f(x) \; - \; f(x).\frac{d}{dx}g(x)}{g(x+0).\,g(x)}$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

Thus, $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$$

This is known as quotient rule of differentiation

Hence proved.

3.3 Application of Theorems on Differentiation

- **Differentiate constant multiple of x^n**

Let us try to understand by solving the following examples:

Example 1. Differentiate the following w.r.t. "x"

(a) $4x^5$

(b) $\frac{5}{2}x^{\frac{2}{5}}$

Solution (a) Let $y = 4x^5$

Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}(4x^5)$$

$$\Rightarrow \frac{dy}{dx} = 4\,.\,\frac{d}{dx}(x^5)$$

$$\Rightarrow \frac{dy}{dx} = 4 \times 5x^{5-1}, \quad [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{dy}{dx} = 20x^4$$

Solution (b) Let $y = \frac{5}{2}x^{\frac{2}{5}}$

Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5}{2}x^{\frac{2}{5}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2}\,.\,\frac{d}{dx}x^{\frac{2}{5}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2} \times \frac{2}{5}x^{\frac{2}{5}-1}$$

$$\Rightarrow \frac{dy}{dx} = x^{-\frac{3}{5}}.$$

- **Differentiate sum (or difference) of functions**

Example 2. Differentiate the following functions w.r.t. 'x'

(a) $y = (2x^2 - 3x + 1) + (4x + 1)$ (b) $y = (2x^3 - 1) - (1 + \frac{1}{x^4})$

Solution (a): given that

$$y = h(x) = (2x^2 - 3x + 1) + (4x - 1)$$

Differentiating w.r.t.'x', we have,

$$\frac{dy}{dx} = \frac{d}{dx}[(2x^2 - 3x + 1) + (4x + 1)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x^2 - 3x + 1) + \frac{d}{dx}(4x + 1)$$

$$\Rightarrow \frac{dy}{dx} = 4x^{2-1} - 3(1) + 0 + 4(1) + 0$$

$$\Rightarrow \frac{dy}{dx} = 4x - 3 + 4$$

$$\Rightarrow \frac{dy}{dx} = h'(x) = 4x + 1$$

Solution (b): Given that

$$y = h(x) = (2x^3 - 1) - \left(1 + \frac{1}{x^4}\right)$$

Differentiating w.r.t. "x", we have

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[(2x^3 - 1) - (1 + x^{-4})\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x^3 - 1) - \frac{d}{dx}(1 + x^{-4})$$

$$\Rightarrow \frac{dy}{dx} = 3\text{x}2x^2 - 0 - 0 - (-4).x^{-5}$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 + 4x^{-5} = 6x^2 + \frac{4}{x^5}$$

- **Differentiate the polynomials**

Example 3. Differentiate w.r.t. "x" the following polynomial functions.

(a) $f(x) = 2x^3 - 4x^2 + 3x + 1$

(b) $f(x) = -5x^3 + 2x^2 + 3x + 5$

Solution (a): Given that

$$f(x) = 2x^3 - 4x^2 + 3x + 1$$

Differentiating w.r.t "x", we have

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(1)$$

$$\Rightarrow f'(x) = 2\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x) + 0$$

$$\Rightarrow f'(x) = 2 \times 3x^2 - 4 \times 2x + 3(1)$$

$$\Rightarrow f'(x) = 6x^2 - 8x + 3$$

Solution (b): Given that;

$$f(x) = -5x^3 + 2x^2 + 3x + 5$$

Differentiating w.r.t "x", we have

$$\frac{d}{dx}f(x) = \frac{d}{dx}(-5x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(5)$$

$$\Rightarrow f'(x) = -5\frac{d}{dx}x^3 + 2\frac{d}{dx}x^2 + 3\frac{d}{dx}(x) + 0$$
$$\Rightarrow f'(x) = -5 \times 3x^2 + 2 \times 2x + 3(1)$$
$$\Rightarrow f'(x) = -15x^2 + 4x + 3$$

- **Differentiate product of functions**

Example 4. Differentiate w.r.t "x" the following product functions using product rule.

(a) $h(x) = (x^2+1)(5x^2+6)$

(b) $h(x) = (x+1)(x+2)(x+3)$

Solution (a): Given that;

$h(x) = (x^2+1)(5x^2+6) = f(x).g(x)$ (say)

Let $f(x) = x^2 + 1$ and $g(x) = 5x^2 + 6$ …(ii)

Differentiating equations both sides of (i) and (ii) w.r.t "x",

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}(x^2+1) \text{ and } \frac{d}{dx}g(x) = \frac{d}{dx}(5x^2+6),$$
$$\Longrightarrow f'(x) = 2x \text{ and } g'(x) = 10x,$$
$$\because \frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x)f'(x),$$
$$\therefore \frac{d}{dx}h(x) = (x^2+1)(10x) + (5x^2+6)(2x),$$
$$\Longrightarrow h'(x) = 10x^3 + 10x + 10x^3 + 12x,$$
$$\Longrightarrow h'(x) = 20x^3 + 22x,$$
$$\Longrightarrow h'(x) = 2x(10x^2 + 11).$$

Solution (b): Given that;

$h(x) = (x+1)(x+2)(x+3)$

$\Longrightarrow h(x) = (x+1)(x^2+5x+6) = f(x).g(x),$ (Say) …(i)

$$\because \frac{d}{dx}h(x) = h'(x) = f(x).g'(x) + g(x).f'(x)$$

Now differentiate both sides of the equations (i) w.r.t "x"

$$\therefore h'(x) = (x+1).\frac{d}{dx}(x^2+5x+6) + (x^2+5x+6).\frac{d}{dx}(x+1),$$
$$\Longrightarrow h'(x) = (x+1).(2x+5+0) + (x^2+5x+6)(1+0),$$
$$\Longrightarrow h'(x) = (x+1).(2x+5) + (x^2+5x+6)(1),$$
$$\Longrightarrow h'(x) = 2x^2 + 7x + 5 + x^2 + 5x + 6,$$
$$\Longrightarrow h'(x) = 3x^2 + 12x + 11.$$

- **Differentiate Quotient of two functions.**

Example 5. Differentiate w.r.t "x" the following quotients (rational) functions using quotient rule;

(a) $h(x)=\frac{x+1}{x^2-2x+3}$ (b) $h(x)=\frac{(x-1)(x+2)}{(x+2)(x+3)}$

Solution (a): Given that

$$h(x)=\frac{x+1}{x^2-2x+3}=\frac{f(x)}{g(x)} \quad \text{Provided } g(x)\neq 0, \forall x\in\mathbb{R} \quad \ldots(i)$$

$$f(x)=x+1 \quad \ldots(ii)$$

$$g(x)=x^2-2x+3 \quad \ldots(iii)$$

Now,

$$h'(x)=\frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2} \quad \ldots(iv)$$

$$f'(x)=\frac{d}{dx}(x+1)=1+0=1 \text{ and } g'(x)=\frac{d}{dx}(x^2+2x+3)=2x-2$$

Now, substitute the values in (iv), we have,

$$h'(x)=\frac{(x^2-2x+3)(1)-(x+1)(2x-2)}{(x^2-2x+3)^2}$$

$$h'(x)=\frac{x^2-2x+3-(2x^2-2x+2x-2)}{(x^2-2x+3)^2}$$

$$h'(x)=\frac{-x^2-2x+5}{(x^2-2x+3)^2}$$

Solution (b): Given that;

$$h(x)=\frac{(x-1)(x+2)}{(x-2)(x+3)},$$

$$\Rightarrow h(x)=\frac{x^2+x-2}{x^2+x-6}=\frac{f(x)}{g(x)} \quad \text{Provided } g(x)\neq 0 \quad \ldots(i)$$

$$\because h'(x)=\left[\frac{f(x)}{g(x)}\right]'=\frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$$

$$\therefore h'(x)=\frac{(x^2+x-6)\frac{d}{dx}(x^2+x-2)-(x^2+x-2)\frac{d}{dx}(x^2+x-6)}{(x^2+x-6)^2}$$

$$\Rightarrow h'(x)=\frac{(x^2+x-6)(2x+1)-(x^2+x-2)(2x+1)}{(x^2+x-6)^2}$$

$$\Rightarrow h'(x)=\frac{2x^3+3x^2-11x-6-(2x^3+3x^2-3x-2)}{(x^2+x-6)^2}$$

$$\Rightarrow h'(x) = \frac{2x^3 + 3x^2 - 11x - 6 - 2x^3 - 3x^2 + 3x + 2}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = \frac{-8x - 4}{(x^2 + x - 6)^2}$$

$$\Rightarrow h'(x) = -\frac{4(2x + 1)}{(x^2 + x - 6)^2}$$

Note: The derivative of an even function is always an odd function and viceversa.
i.e., if $f(-x) = f(x) \Rightarrow f'(-x) = -f'(x)$ and $f(-x) = -f(x) \Rightarrow f'(-x) = f'(x)$.

Exercise 3.3

1. Differentiate the following w.r.t "x".

(i) $5x^5$ (ii) $\frac{7}{9}x^9$ (iii) $-25x^{\frac{-3}{5}}$ (iv) $124\sqrt{x}$

(v) $\frac{1}{22}x^{22}$ (vi) x^{-100} (vii) $15\sqrt[3]{x}$ (viii) $16\sqrt[4]{x^{\frac{3}{4}}}$

(ix) $\frac{-4}{x^{\frac{3}{4}}}$ (x) $\frac{3}{\sqrt[3]{x^2}}$

2. Differentiate the following w.r.t "x".

(i) $\frac{x^5}{a^2+b^2} + \frac{x^2}{a^2-b^2}$ (ii) $2x + \frac{1}{2}x^6$ (iii) $\sqrt[3]{x^2} + \sqrt{x}$

(iv) $\frac{1}{21}x^{21} + \frac{1}{22}x^{22}$ (v) $-\frac{5}{4}x^{\frac{-4}{5}} + \frac{2}{3}x^{\frac{-3}{2}}$

3. Differentiate the following w.r.t "x".

(i) $2ax^3 - \frac{x^2}{b} + 6$ (ii) $x^3 - \frac{3}{7}x^{\frac{7}{3}}$ (iii) $5x^{\frac{3}{5}} - \frac{1}{7}x^{\frac{7}{3}}$

(iv) $x^{10} - 10x^{15}$ (v) $3(\sqrt[3]{x^2}) - 4(\sqrt[4]{x})$

4. Differentiate the following polynomial function w.r.t "x".

(i) $p(x) = x^3 - 3x^2 + 2x + 1$ (ii) $p(x) = x^4 - 3x^2 + 2x - 3$

(iii) $p(x) = x^6 - x^4 + x^3 + x$ (iv) $p(x) = 9x^9 + 7x^7 + \frac{1}{5}x^5 - \frac{1}{4}x^4 + x + 1$

(v) $p(x) = x^3 + x^2 + x + 1$

5. Find the derivative of the following functions using product rule.

(i) $h(x) = (2x - 5).(5x + 7)$ (ii) $h(x) = x.\sqrt{3x^2 + 4}$

(iii) $h(x) = \sqrt[3]{x + 1}.\sqrt[5]{x^2 + 1}$ (iv) $h(x) = x^2(\sqrt{x} + 1)$

(v) $h(x) = (x + 1)^3.x^{\frac{-3}{2}}$

6. Find the derivative of the following functions using quotient rule.

(i) $h(x) = \frac{3x+4}{2x-3}$ (ii) $h(x) = (x^2 - 1).(x^2 + 1)^{-1}$

(iii) $h(x) = \frac{x^2-x+1}{x^2+x-1}$ (iv) $h(x) = \frac{2x^4}{b^2-x^2}$

(v) $h(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

3.4. Chain Rule

The rule for differentiating composite function is called chain rule. In this rule we take the derivative of the outer function and then multiply it with the derivative of the inner function.

The derivative of the composite function fog is $(fog)' = f'(g(x)).g'(x)$ which is called chain rule.

3.4.1 Prove that $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, when $y = f(u)$ and $u = g(x)$. (chain rule)

Proof:

Let $y = f[g(x)] = f(u)$, where $u = g(x)$

$\therefore \delta y = f[g(x+\delta x)] - f[g(x)] = f(u+\delta u) - f(u)$,

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f[g(x+\delta x)] - f[g(x)]}{\delta x}, \text{provided } \delta x \neq 0$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f[g(x+\delta x)] - f[g(x)]}{g(x+\delta x) - g(x)} . \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta u \to 0} \frac{f(u+\delta u) - f(u)}{\delta u} . \lim_{\delta x \to 0} \frac{\delta u}{\delta x},$$

[where $\delta u = g(x+\delta x) - g(x)$ as $\delta x \to 0$, $\delta u \to 0$]

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du} f(u) . \frac{d}{dx} g(x) = \frac{dy}{du} . \frac{du}{dx}$$ [$\because y = f(u)$ and $u = g(x)$]

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

Hence proved.

Example: Differentiate $y = (5x-4)^5$.

Solution:

$y = (5x^2-4)^5$ and let $u = 5x^2 - 4$

then $y = u^5$ ...(i)

$u = 5x^2 - 4$ ...(ii)

Differentiating equation (i) w.r.t "u" and equation (ii) w.r.t "x"

$$\therefore \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = 2(5)x - 0 = 10x$$

By chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = 5u^4 . 10x = 50x\,(5x^2 - 4)^4$$

3.4.2 Show that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Proof: If $y = f(x)$ is any differentiable function in the domain of x, then its inverse function is defined as $x = g(y)$, such that:

$$(gof)(x) = g(f(x)) = g(y) = x \qquad \ldots(i)$$

Differentiating both sides of equation (i) w.r.t x

$$g'(y).f'(x) = 1 \qquad \text{(by chain rule)}$$

$$\Rightarrow f'(x) = \frac{1}{g'(y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left[\because f'(x) = \frac{dy}{dx} \text{ and } g'(y) = \frac{dx}{dy}\right]$$

Hence Showed.

3.4.3 Use chain rule to show that $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$.

Let $\quad y = [f(x)]^n, \quad \forall x \in \mathbb{R}$

Suppose that

$$u = f(x) \qquad \ldots (i)$$

then $\quad y = u^n \qquad \ldots (ii)$

From equation (i) and (ii) by differentiating

$$\therefore \frac{du}{dx} = \frac{d}{dx} f(x) = f'(x)$$

and $\quad \frac{dy}{du} = \frac{d}{du}(u^n)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du}(u^n).\frac{d}{dx}(u) = nu^{n-1}.f'(x)$$

$$\Rightarrow \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$$

Hence Showed.

Example: Differentiate $(x^4 - 4x^2 + 5)^{\frac{5}{2}}$.

Solution: Let $y = (x^4 - 4x^2 + 5)^{\frac{5}{2}}$

Here $f(x) = x^4 - 4x^2 + 5$ and $n = \frac{5}{2}$.

$$\because \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$$

$$\therefore \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{5}{2}(x^4 - 4x^2 + 5)^{\frac{5}{2}-1}.\frac{d}{dx}(x^4 - 4x^2 + 5)$$

$$\Rightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{5}{2}(x^4 - 4x^2 + 5)^{\frac{3}{2}}.(4x^3 - 8x + 0)$$

$$\Rightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = \frac{20}{2}.(x^4 - 4x^2 + 5)^{\frac{3}{2}}(x^3 - 2x)$$

$$\Rightarrow \frac{d}{dx}(x^4 - 4x^2 + 5)^{\frac{5}{2}} = 10x(x^2 - 2)(x^4 - 4x^2 + 5)^{\frac{3}{2}}$$

3.4.4 Find the derivative of implicit functions

The chain rule will help us to find the derivative of implicit functions.

Example 1. Find $\frac{dy}{dx}$, if $x^2 + y^2 + 2gx + 2fy + c = 0$.

Solution: Given that

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiating both the sides w.r.t. "x", keeping y as a function of x ,

$$\therefore \frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Rightarrow 2x + 2y.\frac{dy}{dx} + 2g(1) + 2f.\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow (y + f).\frac{dy}{dx} = -(x + g)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{(x + g)}{(y + f)} = -\frac{x + g}{y + f}}$$

Example 2. Find $\frac{dy}{dx}$, if $x^2y + 2y^3 = 3x + 2y$

Solution: Given that

$$x^2y + 2y^3 = 3x + 2y$$

It is an implicit equation

$\therefore$ diff: both the sides w.r.t. "x" regarding y as function of x,

$$\therefore \frac{d}{dx}(x^2y + 2y^3) = \frac{d}{dx}(3x + 2y)$$

$$\Rightarrow \frac{d}{dx}(x^2y) + \frac{d}{dx}(2y^3) = \frac{d}{dx}(3x) + \frac{d}{dx}(2y)$$

$$\Rightarrow x^2.\frac{dy}{dx} + y.\frac{d}{dx}(x^2) + 3 \times 2y^2.\frac{dy}{dx} = 3(1) + 2\frac{dy}{dx}$$

$$\Rightarrow x^2.\frac{dy}{dx} + 6y^2\frac{dy}{dx} - 2\frac{dy}{dx} = 3 - 2xy$$

$$\Rightarrow (x^2 + 6y^2 - 2)\frac{dy}{dx} = 3 - 2xy$$

$$\frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 6y^2 - 2}$$

3.5 Differentiation of trigonometric and Inverse Trigonometric Functions

3.5.1 Differentiate Trigonometric functions ($\sin x, \cos x, \tan x, \csc x, \sec x$ *and* $\cot x$) from the first principle

In the process of finding the derivative of trigonometric functions, we assume that x is measured in radians.

- **Differentiate $\sin x$ from the first principle.**

Consider the sine function $s: \mathbb{R} \to \mathbb{R}$, where $s(x) = \sin x, \forall x \in \mathbb{R}$.

Let $s(x) = \sin x$

$$\therefore \quad y + \delta y = s(x + \delta x) = \sin(x + \delta x)$$

Using the first principle, i.e.,

$$\therefore \quad s'(x) = \lim_{\delta x \to 0} \frac{s(x+\delta x) - s(x)}{\delta x}$$

$$\therefore \quad s'(x) = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin(x)}{\delta x}$$

$$\Rightarrow \quad s'(x) = \lim_{\delta x \to 0} \frac{2\cos\left(\frac{x + \delta x + x}{2}\right) \cdot \sin\left(\frac{x + \delta x - x}{2}\right)}{\delta x}$$

$$\left[\because \sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right]$$

$$\Rightarrow \quad s'(x) = \lim_{\frac{\delta x}{2} \to 0} 2\cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{2 \cdot \frac{\delta x}{2}}$$

$$\Rightarrow \quad s'(x) = \lim_{\frac{\delta x}{2} \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$\Rightarrow \quad s'(x) = \cos(x+0)\cdot 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \quad s'(x) = \cos x$$

Thus, $\frac{d}{dx}(\sin x) = \cos x$

> **Note:** • $\frac{d}{dx}\sin ax = \cos ax.\frac{d}{dx}(ax) = a\cos ax$
>
> • $\frac{d}{dx}\sin^n x = n\sin^{n-1} x\cdot\frac{d}{dx}(\sin x) = n\sin^{n-1} x.\cos x$

- **Differentiate $\tan x$ from first Principle**

Let $t(x) = \tan x$

Using first principle,

i.e., $f'(x) = \lim_{\delta x\to 0}\frac{t(x+\delta x)-t(x)}{\delta x}$

$$\therefore \quad t'(x) = \lim_{\delta x\to 0}\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]\cdot\frac{1}{\delta x}$$

$$\Rightarrow \quad t'(x) \lim_{\delta x\to 0}\left[\frac{\sin(x+\delta x).\cos x - \cos(x+\delta x).\sin x}{\delta x\cos(x+\delta x).\cos x}\right]$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\frac{\sin(x+\delta x - x)}{\delta x}\cdot\frac{1}{\lim_{\delta x\to 0}\cos(x+\delta x)\cdot\cos x}$$

$$\Rightarrow \quad t'(x) = \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\frac{1}{\lim_{\delta x\to 0}\cos(x+\delta x)\cdot\cos x}$$

$$\Rightarrow \quad t'(x) = 1\cdot\frac{1}{\cos(x+0)\cos x} \qquad \because \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\Rightarrow \quad t'(x) = \frac{1}{\cos x\cdot\cos x} = \frac{1}{\cos^2 x} = \sec^2 x \qquad [\text{Provided } \cos x \neq 0]$$

Thus, $\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$

> **Note:** • $\frac{d}{dx}(\tan ax) = \sec^2 ax\cdot\frac{d}{dx}(ax) = a\sec^2 ax$
>
> • $\frac{d}{dx}(\tan^n x) = n\tan^{n-1} x\cdot\frac{d}{dx}(\tan x) = n\tan_x^{n-1}\cdot\sec^2 x$

- **Differentiate $\sec x$ from first Principle.**

Let $y = \sec x$

$$\therefore \quad y + \delta y = \sec(x+\delta x)$$

$$\because \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec(x)}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}\right] \cdot \frac{1}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x) \cdot \cos x \cdot \delta x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{2\sin\left(\frac{x+x+\delta x}{2}\right) \cdot \sin\left(\frac{x+\delta x - x}{2}\right)}{\cos(x+\delta x) \cdot \cos x \cdot \delta x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{2\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x) \cdot \cos x}\right] \cdot \lim_{\frac{\delta x}{2} \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{2 \cdot \frac{\delta x}{2}}$$

$$\left[\because \cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin(x+0)}{\cos(x+0) \cdot \cos x} \cdot \frac{1}{2}\, 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \tan x \,.\sec x$$

$$\Rightarrow \frac{dy}{dx} = \sec x \,.\tan x\,,$$

Thus, $\boxed{\frac{d}{dx}(\sec x) = \sec x \,.\tan x}$

In general cases:

- $\frac{d}{dx}(\sec ax) = \sec ax \,.\tan ax \,.\frac{d}{dx}(ax)$

  $= a \sec ax \,.\tan ax$
- $\frac{d}{dx}(\sec^n x) = n \sec^{n-1} x \,.\frac{d}{dx}(\sec x)$

  $= n \sec^{n-1} x \,.\sec x \,.\tan x$

  $= n \sec^n x \,.\tan x$

Note: The derivative of cosine, cosecant and cotangent are left as an exercise for readers.

Examples 1. Differentiate "x" $\cos\sqrt{x}$ by ab-intio/first principle.

Solution: Let $y = f(x) = \cos\sqrt{x}$

$$\therefore \quad y + \delta y = f(x+\delta x) = \cos\sqrt{x+\delta x},$$

$$\because \quad \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\therefore \quad \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\cos\sqrt{x+\delta x} - \cos\sqrt{x}}{\delta x}$$

Using trigonometric formula

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right).\ \sin\left(\frac{a-b}{2}\right)$$

we have,

$$\Rightarrow \frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\ \sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -2\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\delta x \to 0} \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{x+\delta x - x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -2\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\delta x \to 0} \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{2\left[\left(\sqrt{x+\delta x}\right)^2-\left(\sqrt{x}\right)^2\right]}{2}}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right).\lim_{\theta \to 0}\frac{\sin\theta}{\theta}.\frac{1}{\lim_{\delta x \to 0}\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\left[\text{where } \theta = \frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)}{2}\right], \text{as } \delta x \to 0, \theta \to 0$$

$$= -\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right).1.\frac{1}{\left(\sqrt{x+0}+\sqrt{x}\right)} \qquad \left[\because \lim_{\delta x \to 0}\frac{\sin x}{x} = 1\right]$$

$$= -\sin\left(\frac{2\sqrt{x}}{2}\right).\frac{1}{2\sqrt{x}}$$

$$= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}},$$

Thus, $\boxed{\dfrac{d}{dx}(\cos\sqrt{x}) = \dfrac{-\sin\sqrt{x}}{2\sqrt{x}}}$

Example 2. Differentiate:

(i) $y = \frac{x^2+\tan x}{3x+2\tan x}$ w.r.t x (ii) $y = \cos^2 x$ w.r.t $\sin^2 x$

Solution (i): Given that

$$y = \frac{x^2 + \tan x}{3x + 2\tan x}$$

Differentiating w.r.t "x" by using quotient rule, we have

$$\therefore \quad \frac{dy}{dx} = \frac{(3x + 2\tan x).\frac{d}{dx}(x^2 + \tan x) - (x^2 + \tan x).\frac{d}{dx}(3x + 2\tan x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(3x + 2\tan x)(2x + \sec^2 x) - (x^2 + \tan x)(3 + 2\sec^2 x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x^2 + 3x\sec^2 x + 4x\tan x + 2\tan x\sec^2 x - (3x^2 + 2x^2\sec^2 x + 3\tan x + 2\tan x.\sec^2 x)}{(3x + 2\tan x)^2}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{3x^2 + (4x - 3)\tan x + x(3 - 2x)\sec^2 x}{(3x + 2\tan x)^2}}$$

Solution (ii): Given that:

$$y = \cos^2 x, \qquad \ldots\text{(i)}$$

and let

$$u = \sin^2 x, \qquad \ldots\text{(ii)}$$

In this case, we have to find $\frac{dy}{du}$

Here,

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos^2 x) = 2\cos x.\frac{d}{dx}(\cos x) = -2\sin x.\cos x$$

and $\frac{du}{dx} = \frac{d}{dx}(\sin^2 x) = 2\sin x.\frac{d}{dx}(\sin x) = 2\sin x.\cos x$

Using chain rule:

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = \frac{dy}{dx} \div \frac{du}{dx}$$

$$\therefore \frac{dy}{du} = \frac{-2\sin x\ \cos x}{2\sin x\cos x}$$

$$\Rightarrow \boxed{\frac{dy}{du} = -1}$$ Provided $\sin x \neq 0$ and $\cos x \neq 0$

3.5.2 Differentiate inverse trigonometric functions ($arc\ \sin x$, $arc\cos x$, $arc\tan x$, $arc\ \csc x$, $arc\sec x$ and $arc\ \cot x$) using differentiation formulae.

- **Differentiate arc $\sin x$ or ($\sin^{-1} x$)**

Let $\quad y = \sin^{-1} x \qquad \forall\, x \in (-1, 1)$

$\Rightarrow \sin y = x$

$\therefore$ Differentiating w.r.t "x", regarding y as a function of x, we have

$$\therefore \frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\sin y).\frac{dy}{dx} = 1$$

$$\Rightarrow \cos y.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \qquad \text{[Provided } \cos y \neq 0]$$

$$\therefore \qquad \cos y = \pm\sqrt{1-\sin^2 y}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$$

The principal domain of $\sin y$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which $\cos y$ is +ve.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad (\because x = \sin y)$$

Thus, $\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)}$

Note: $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{a}{a\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-x^2}}$

- **Differentiate $arc\tan$ or $\tan^{-1} x$**

Let $\quad y = \tan^{-1} x \qquad \forall x \in \mathbb{R}$

$\Longrightarrow \quad \tan y = x$

Differentiating w.r.t "x" regarding y as a function of x,

$$\therefore \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} \qquad \forall y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} \qquad [\because \sec^2 y = 1 + \tan^2 y]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$

Thus, $\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}}$

Note: $\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{a^2}{(a^2+x^2)} \cdot \frac{1}{a} = \frac{a}{a^2+x^2}$

- **Differentiate $arc\sec x, \forall x \in \mathbb{R} - [-1, 1]$ $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$**

Let $y = sec^{-1}x, \forall x \in \mathbb{R} - [-1,1] \longrightarrow \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

then, $x = sec\, y, \quad \forall y \in [0, \pi]$ and $y \neq \frac{\pi}{2}$,

It is an implicit equation:

Differentiating w.r.t "x" regarding y as function of x

$$\therefore \frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$$

$$\Rightarrow 1 = \frac{d}{dy}(\sec y) \cdot \frac{d}{dx}(y)$$

$$\Rightarrow 1 = \sec y \cdot tan\ y \cdot \frac{dy}{dx}$$

or

$$\sec y \cdot tan\ y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{sec\ y \cdot tan\ y} \qquad \forall y \in (0, \pi) \text{ and } y \neq \frac{\pi}{2}$$

when, $y \in \left(0, \frac{\pi}{2}\right)$, $\sec y$ and $\tan y$ are positive, so that $\sec y = x$, i.e., x is positive in this case,

and $tan\ y = \sqrt{sec^2\ y - 1} = \sqrt{x^2 - 1}$,

Thus, $\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}, \text{ when } x > 0}$

When, $y \in \left(\frac{\pi}{2}, \pi\right)$, $\sec y$ and $\tan y$ are negative so that x is negative and in this case.

$\tan y = -\sqrt{x^2 - 1}$, when $x < 0$,

Thus,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{(-x)(-\sqrt{x^2-1})} = \frac{1}{x\sqrt{x^2-1}} \quad \text{... (ii)}$$

Combining equations (i) and (ii) we have,

Thus, $$\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}},} \quad \forall x \in \mathbb{R} - [-1,1]$$

Note: The derivatives of $cos^{-1}x, csc^{-1}x$ and $cot^{-1}x$ are left as an exercise for readers.

- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example 1. If $y = x\, sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}$, find $\frac{dy}{dx}$

Solution: Given that

$$y = x\, sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$$

diff: w.r.t "x" we have,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x\, sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x\, sin^{-1}\frac{x}{a}\right) + \frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = x\frac{d}{dx}\left(sin^{-1}\frac{x}{a}\right) + sin^{-1}\left(\frac{x}{a}\right).\frac{d}{dx}(x) + \frac{1}{2\sqrt{a^2-x^2}}.\frac{d}{dx}(a^2 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = x.\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{d}{dx}\left(\frac{x}{a}\right) + sin^{-1}\left(\frac{x}{a}\right) \times 1 + \frac{0 - 2x}{2\sqrt{a^2-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x \times a}{\sqrt{a^2-x^2}} \times \frac{1}{a} + sin^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2-x^2}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = sin^{-1}\left(\frac{x}{a}\right)}$$

Example 2. If $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$, find $\frac{dy}{dx}$.

Solution: Given that:

$$y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

Differentiating w.r.t 'x'

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right),$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right), \qquad \left[\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+1)}{\sqrt{(x^2+1)^2-(x^2-1)^2}} \times \frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x^2+1} \times \frac{(x^2+1)(2x)-(x^2-1)(2x)}{\sqrt{x^4+2x^{2+}1-x^4+2x^2-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x^2+1} \times \frac{2x^3+2x-2x^3+2x}{\sqrt{4x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4x}{2x(x^2+1)} \qquad \text{provided } x \neq 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2}{(x^2+1)}}$$

Exercise 3.4

1. Find the derivatives of the following using chain rule.

   (i) $y = (x^4+5x^2+6)^{\frac{3}{2}}$ (ii) $y = \left(\frac{x-1}{x+1}\right)^{\frac{3}{4}}$ (iii) $y = \sqrt{\frac{2+x}{3+x}}$

   (iv) $y = \left(x+\sqrt{x^2-1}\right)^n$ (v) $y = \sqrt[3]{\frac{x^3+1}{x^3-1}}$

2. Differentiate $\frac{x^3}{1+x^3}$ w.r.t x^3.

3. If f is a function with $y = f(x)$, given implicitly, find $\frac{dy}{dx}$, where it exists in the following cases.

   (i) $y - xy - \sin y = 0$ (ii) $y^3 - 3y + 2x = 0$

(iii) $x^2 + y^2 + 4x + 6y - 12 = 0$ (iv) $\sin x\, y + \sec x = 2$
(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$ (vi) $y(x^2+1) = x(y^2+1)$

4. If $\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$, where a and b are non-zero constants, find $\dfrac{du}{dv}$ and $\dfrac{dv}{du}$.

5. Find the slope of the tangent to the curve $3x^2 - 7y^2 + 14y - 27 = 0$ at the point $(-3,0)$.

6. Differentiate the following by using first principle method.
(i) $\sin 4x$ (ii) $\cos^2 2x$ (iii) $\sec\sqrt{x}$
(iv) $\sqrt{\tan x}$ (v) $\csc 3x$ (vi) $\cot 2x$

7. Using differentiations rules, differentiate w.r.t. their involved variables:
(i) $f(x) = (x+2)\,.\sin x$ (ii) $f(\theta) = \tan^2\theta\,.\sec^3\theta$
(iii) $f(t) = \sin^2 3t\,.\,\cos^3 t$ (iv) $f(x) = \sqrt{\dfrac{\sin 2x}{\cos x}}$
(v) $f(\theta) = \dfrac{\tan\theta - 1}{\sec\theta}$ (vi) $f(x) = \sin x^2 + \sin^2 x$

8. Differentiate $\dfrac{1+\tan^2 x}{1-\tan^2 x}$ w.r.t. $\tan^2 x$.

9. Find $\dfrac{dy}{dx}$ of the following:
(i) $y = \sin^{-1}\sqrt{\dfrac{1-\cos x}{2}}$ (ii) $y = \cot^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$
(iii) $y = \tan^{-1}\left(\dfrac{\sin 2x}{1+\cos 2x}\right)$ (iv) $y = x + \cos^{-1}x\,.\sqrt{1-x^2}$
(v) $y = \dfrac{x\tan^{-1}x}{1+x^2}$ (vi) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

10. If $y = \tan^{-1}\left(2\tan\frac{x}{2}\right)$, then prove that $\dfrac{dy}{dx} = 4\left(\dfrac{1+y^2}{4+x^2}\right)$

11. If $\dfrac{y}{x} = \tan^{-1}\left(\dfrac{x}{y}\right)$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

12. If $y = \tan(a\tan^{-1}x)$, show that $(1+x^2)\dfrac{dy}{dx} - a(1+y^2) = 0$.

13. Find $\dfrac{dy}{dx}$
(i) $x = a\sin\theta$ and $y = a\cos\theta$ (ii) $x = t + \frac{1}{t}$ and $y = t + 1$

(iii) $x = \frac{a(1-t^2)}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$, (a and b are constant)

(iv) $x = a\theta^2$ and $y = 2a\theta$, (a is constant)

14. If $y = \sqrt{tanx + \sqrt{tanx + \sqrt{tanx} + .. + \infty}}$, prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

15. Find the derivatives of $\cos^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$ by using differentiation formula.

3.6 Differentiation of Exponential and Logarithmic Functions

3.6.1 Find the derivatives of e^x and a^x from first principle

(a) Derivative of e^x

Let $y = f(x) = e^x$

$$\therefore y + \delta y = f(x+\delta x) = e^{x+\delta x}$$

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$$

$$= e^x \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} \qquad \left(\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right)$$

$$= e^x . 1 = e^x$$

Thus, $\Rightarrow \frac{d}{dx}(e^x) = e^x$

Note: $\frac{d}{dx}(e^{ax}) = e^{ax}.\frac{d}{dx}(ax) = ae^{ax}$

(b) Derivatives of $a^x, \forall\, a > 0$ and $a \neq 1$

Let $y = f(x) = a^x, \forall a > 0$ and $x \in \mathbb{R}$

$$\therefore y + \delta y = f(x+\delta x) = a^{x+\delta x},$$

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$, Provided $\delta x \neq 0$ and its limit exists

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x},$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} a^x \left(\frac{a^{\delta x} - 1}{\delta x}\right),$$

$$\Rightarrow \frac{dy}{dx} = a^x . \lim_{\delta x \to 0} \left(\frac{a^{\delta x} - 1}{\delta x}\right),$$

$$\Rightarrow \frac{dy}{dx} = a^x . \ln a, \qquad \left(\because \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a\right)$$

Thus, $\Rightarrow$ $\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$

Note: $\frac{d}{dx}\left(a^{bx}\right) = a^{bx} . b \ln a\, (bx) = b a^x \ln a$

Example 1. Find $\frac{dy}{dx}$ when $y = e^{\sin x} + a^{\cos x}$

Solution: Given that

$$y = e^{\sin x} + a^{\cos x}$$

diff: w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x} + a^{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin x}\right) + \frac{d}{dx}\left(a^{\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x} \frac{d}{dx}(\sin x) + a^{\cos x} \ln a \frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x} \cos x + a^{\cos x} \ln a\,(-\sin x)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = e^{\sin x} \cos x - a^{\cos x} \ln a \sin x}$$

3.6.2 Find the derivative of $\ln x$ and $\log_a x$ from first principle

- **Derivative of $\ln x$ from first principle**

Let $y = f(x) = \ln x\ , \ \forall\ x > 0$

$$\therefore y + \delta y = f(x + \delta x) = \ln(x + \delta x)$$

$$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{ln(x + \delta x) - ln(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\ln\left(\frac{x + \delta x}{x}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{x} . \frac{x}{\delta x} . \ln\left(1 + \frac{\delta x}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \qquad \left[\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \times 1 = \frac{1}{x} \qquad [\because \ln e = 1]$$

Thus, $\frac{d}{dx}(\ln x) = \frac{1}{x}$, $\forall \; x > 0$ ...(i)

Similarly, $\frac{d}{dx}\ln(-x) = \frac{1}{-x} \times (-1) = \frac{1}{x}$, $\forall x < 0$ ...(ii)

Combining (i) and (ii), we have

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Note: $\frac{d}{dx}(\ln a\,x) = \frac{1}{ax} \cdot \frac{d}{dx}(ax) = \frac{1}{x}$, $\forall \; x > 0$

- **Derivative of $\log_a x$ by first principle**

Let $y = f(x) = \log_a x$

$\therefore$ $y + \delta y = f(x + \delta x) = \log_a(x + \delta x)$

$\because \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$, (Provided $\delta x \neq 0$ and limit exist)

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\log_a(x+\delta x) - \log_a x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{x} \cdot \frac{x}{\delta x} \log_a\left(\frac{x+\delta x}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \lim_{\delta x \to 0} \log_a\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \log_a\left[\lim_{\delta x \to 0} \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} \log_a e \; , \qquad \left[\because \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x \ln a} , \qquad \left[\because \log_a e = \frac{1}{\ln a}\right]$$

Thus, $\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$

Note: $\frac{d}{dx}(\log_a bx) = \frac{1}{bx \ln a} \cdot \frac{d}{dx}(bx) = \frac{1}{x \ln x}$, $\forall \; x > 0$

Example 1. Find $\frac{dy}{dx}$, when: $y = \ln(x^2+4)$

Solution: Given that

$$y = \ln(x^2+4)$$

Differentiating w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\ln(x^2+4), \qquad \left[\because \frac{d}{dx}\ln x = \frac{1}{x}\right]$$

$$\frac{dy}{dx} = \frac{1}{x^2+4}\cdot\frac{d}{dx}(x^2+4) = \frac{2x}{x^2+4}$$

Example 2. Find $\frac{dy}{dx}$, when: $y = \log_{10}\sqrt{x^2+2x} - 4x^4$

Solution: Given that

$$y = \log_{10}\sqrt{x^2+2x} - 4x^4$$

$$\Rightarrow \quad y = \log_e\sqrt{x^2+2x}.\log_{10}e - 4x^4 \quad [\because \log_{10}^{x} = \log_e^{x}.\log_{10}^{e} \text{ and } \log_e^{x} = \ln x]$$

$$\Rightarrow \quad y = \ln\left(\sqrt{x^2+2x}\right).\log_{10}e - 4x^4$$

$$y = \frac{\log_{10}e}{2}.\ln(x^2+2x) - 4x^4$$

Differentiating w.r.t "x"

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log_{10}e}{2}.\ln(x^2+2x) - 4x^4\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log_{10}e}{2}.\frac{d}{dx}\ln(x^2+2x) - 4\frac{d}{dx}(x^4)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\ln 10}.\frac{1}{x^2+2x}\frac{d}{dx}(x^2+2x) - 4\times 4x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln 100}\times\frac{(2x+2\times 1)}{x^2+2x} - 16x^3$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{(x+1)}{x(x+2)\ln 100} - 16x^3}$$

3.6.3 Use logarithmic differentiation to find derivative of algebraic expression involving product, quotient and power.

Example 1. Differentiate $\left[\frac{x(x+1)}{(x+2)}\right]$, w.r.t. "$x$"

Solution: Let $y = \ln\frac{x(x+1)}{(x+2)}$

Taking ln on both sides

$$\ln y = \ln\left(\frac{x(x+1)}{(x+2)}\right)$$

$$\ln y = \ln x + \ln(x+1) - \ln(x+2)$$

Differentiating w.r.t x

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+1} - \frac{1}{x+2}$$

$$\frac{dy}{dx} = \frac{x(x+1)}{(x+2)}\left[\frac{(x+1)(x+2) + x(x+2) - x(x+1)}{x(x+1)(x+2)}\right]$$

$$\frac{dy}{dx} = \frac{x(x+1)}{(x+2)}\frac{(x^2+3x+2+x^2+2x-x^2-x)}{x(x+1)(x+2)}$$

$$\frac{dy}{dx} = \frac{x^2+4x+2}{(x+2)^2}$$

Example 2. If $x^y = y^x$, find $\frac{dy}{dx}$.

Solution: $x^y = y^x$ Given that

Taking natural logarithm of both sides, we have

$\therefore \quad \ln x^y = \ln y^x$

$\Rightarrow \quad y \ln x = x \ln y \qquad [\therefore \ln a^x = x \ln a]$

Differentiating both sides w.r.t. "x" regarding y as a function of x.

$$\therefore \frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y)$$

$$\Rightarrow \quad y\frac{d}{dx}(\ln x) + \ln x\,.\frac{d}{dx}(y) = x.\frac{d}{dx}(\ln y) + \ln y\,.\frac{d}{dx}(x)$$

$$\Rightarrow \frac{y}{x} + \ln x\;.\frac{dy}{dx} = x.\frac{d}{dy}\;(\ln y).\frac{dy}{dx}\; + \;\ln y\; \times 1 \qquad \left[\because \frac{d}{dx}(\ln x) = \frac{1}{x}\right]$$

$$\Rightarrow \ln x\,.\frac{dy}{dx} - \frac{x}{y}\;.\frac{dy}{dx} = \left(\ln y - \frac{y}{x}\right)$$

$$\Rightarrow \left(lnx - \frac{x}{y}\right)\frac{dy}{dx} = \left(\ln y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\ln y - \frac{y}{x})}{(lnx - \frac{x}{y})}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{y(x\ln y - y)}{x(y\ln x - x)}}$$

List of Derivatives of the basic standard functions shown in the list 1.

Sr No.	$y = f(x)$	Derivative $\frac{dy}{dx}$
1.	$y = \sin x$	$\cos x$
2.	$y = \cos x$	$-\sin x$
3.	$y = \tan x$	$\sec^2 x$
4.	$y = \sec x$	$\sec x \tan x$
5.	$y = \cot x$	$-\operatorname{cosec}^2 x$
6.	$y = \operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
7.	$y = \sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
8.	$y = \cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
9.	$y = \tan^{-1} x$	$\frac{1}{1+x^2}$
10.	$y = \cot^{-1} x$	$-\frac{1}{1+x^2}$
11.	$y = \csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
12.	$y = \sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
13.	$y = \log_a x$	$\frac{1}{x \ln a}, \quad \forall a > 0$
14.	$y = \ln x$	$\frac{1}{x}, \quad \forall x > 0$

Exercise 3.5

1. Differentiate the following w.r.t. "x"

(i) $x^2 + 2^x$ (ii) $4^x + 5^x$ (iii) $e^{\tan x + \cot x}$

(iv) $e^{\tan x^2}$ (v) $e^{2\ln(2x+1)}$ (vi) $\log_{10} x$

(vii) $\frac{e^x}{x^2+1}$ (viii) $x^2 + 2^x + a^{2x}$ (ix) $(\ln x)^x$

(x) $\ln\left(\sqrt{e^{3x} + e^{-3x}}\right)$ (xi) $\ln(\sin(\ln x))$ (xii) $\ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$

2. Using logarithmic differentiation to find $\frac{dy}{dx}$ if

(i) $y=\sqrt{\frac{x^2-1}{x^2+1}}$ (ii) $y=x^3\sqrt{x}$ (iii) $y=xe^{\cos x}$

(iv) $y=e^{-2x}(x^2+2x+1)$ (v) $y=\ln\left(\frac{e^x}{1+e^x}\right)$ (vi) $y=\sqrt{\frac{1+e^x}{1-e^x}}$

3. Find $\frac{dy}{dx}$ if

(i) $y=\frac{1-x^2}{\sqrt{1+x^2}}$ (ii) $y=\sqrt{\frac{1-x}{1+x}}$

4. Find $\frac{dy}{dx}$ if

(i) $y=x^{\sin x}$ (ii) $y=(\sin^{-1}x)^{\ln x}$ (iii) $y=(\tan^{-1}x)^{\sin x+\cos x}$

(iv) $y=(\ln x)^{cosx}$ (v) $y=x^x$ (vi) $y=\ln\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}\right)$

5. Find $\frac{dy}{dx}$, when:

(i) $x^y.y^x=1$ (ii) $\ln(xy)=x^2+y^2$

(iii) $y=\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$ (iv) $y=x^y$

(v) $y=\cos x\ln(\sin^{-1}x)$ (vi) $x^n.y^n=a^n$

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

3.7.1 (a) Differentiation of hyperbolic functions

(i) Differentiate $\sinh x$ by first principle

Let $y=f(x)=\sinh x=\frac{e^x-e^{-x}}{2}$

Differentiating w.r.t. "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x}{2}\right)-\frac{d}{dx}\left(\frac{e^{-x}}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{e^x}{2}\times 1-\frac{e^{-x}}{2}\times(-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Thus, $\boxed{\frac{d}{dx}(\sinh x) = \cosh x}$

$$\boxed{\frac{d}{dx}(\sinh ax) = a \cos h\, ax}$$

Derivatives of $\cosh x, \tanh x, \operatorname{csch} x, \operatorname{sech} x$ *and* $\coth x$ are explained below:

- $$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$
$$= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2}$$
$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^2 + e^{-x}}\right)^2 = \sec h^2 x$$

- $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = \frac{(e^x+e^{-x})(0) - 2(e^x + e^{-x} \times -1)}{(e^x+e^{-x})^2}$$
$$= \frac{-2}{(e^x + e^{-x})} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$= -\sec hx \,.\, \tan hx$$

Derivative of $\cosh x$, cosech x and coth x are left as an exercise for the readers.

Thus, we have the list of derivatives of hyperbolic functions.

• $\frac{d}{dx}(\sinh x) = \cosh x$	• $\frac{d}{dx}(\cosh x) = \sinh x$
• $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$	• $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \,.\, \tanh x$
• $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	• $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Example 1. Differentiate the following w.r.t. "x"

(i) $x \cosh 2x$ (ii) $\frac{csch\, 2x}{x^2}$

Solution (i): Let $y = x \cosh 2x$

Differentiating w.r.t x, we have

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}(x \cosh 2x)$$

$$\frac{dy}{dx} = x\frac{d}{dx}(\cosh 2x) + \cosh 2x \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x \times \sinh 2x \frac{d}{dx}(2x) + \cosh 2x \times 1$$

$$\frac{dy}{dx} = 2x \sinh 2x + \cosh 2x$$

Thus, $\frac{d}{dx}\left(x \cosh 2x\right) = 2x \sinh 2x + \cosh 2x$

Solution (ii): Let $y = \frac{\operatorname{csch} 2x}{x^2}$

Differentiating w.r.t x, we have

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\operatorname{csch} 2x}{x^2}\right)$$

$$\frac{dy}{dx} = \left[x^2 . \frac{d}{dx}(\operatorname{csch} 2x) - \operatorname{csch} 2x \frac{d}{dx}(x^2)\right] \cdot \frac{1}{(x^2)^2}$$

$$\frac{dy}{dx} = \left[x^2 - \operatorname{csch} 2x . \coth 2x \frac{d}{dx}(2x) - \operatorname{csch} 2x . 2x\right] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = [-2x^2 \operatorname{csch} 2x \coth 2x - 2x \operatorname{csch} 2x] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = [-2x \operatorname{csch} 2x\,(x \coth 2x - 1)] \cdot \frac{1}{x^4}$$

$$\frac{dy}{dx} = \frac{-2 \operatorname{csch} 2x\,(x \coth 2x - 1)}{x^3}$$

Thus, $\frac{d}{dx}\left(\frac{\operatorname{csch} 2x}{x^2}\right) = \frac{-2 \operatorname{csch} 2x(x \coth 2x - 1)}{x^3}$

- **Differentiation of inverse hyperbolic functions**
  $\sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x, \operatorname{csch}^{-1} x, \operatorname{sech}^{-1} x, \coth^{-1} x$
- Find derivative of $\sinh^{-1} x$, w.r.t "x"

Solution: Let $y = \sinh^{-1} x \quad , \quad \forall x \in \mathbb{R}$

then $\sinh y = x, \quad \forall \ y \in \mathbb{R}$

Differentiating w.r.t. "x" both sides, regarding y as a function of x

$$\therefore \frac{d}{dx}(\sinh y) = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\sin hy) . \frac{dy}{dx} = 1$$

$$\Rightarrow \cos hy . \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos hy} \qquad (\because \cosh y > 0)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+\sin h^2 y}} \qquad [\because \cosh y = \sqrt{1+\sin h^2 y}]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad , \qquad [\because \sinh y = x]$$

Thus, $\boxed{\frac{dy}{dx}(\sin h^{-1} x) = \frac{1}{\sqrt{1+x^2}}, \quad \forall\ x \in \mathbb{R}}$

(iii) Find the derivative of $\tanh^{-1}x$, w.r.t. "x".

Solution: Let $y = \tan h^{-1}x, \forall\ x \in (-1,1)$

Then $\tanh y = x, \qquad \forall\ y\ \in\ \mathbb{R}$

diff: w.r.t. "x" both sides, keeping y as a function of x

$$\therefore \frac{d}{dx}(\tan hy)\frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\tan hy)\frac{dy}{dx} = 1$$

$$\Rightarrow \sec h^2 y . \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{sec^2 hy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\ \tanh^2 y} \qquad [\because\ \text{sech}^2 y = 1 - \tanh^2 y]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}, \forall\ x\ \in (-1,1)$$

Thus, $\boxed{\frac{d}{dx}(\ \tanh^{-1} x) = \frac{1}{1-x^2}, \forall\ x\ \in (-1,1)}$

(iv) Find the derivative of $\text{csch}^{-1}x$.

Solution: Let $y\ = \text{csch}^{-1}x\ , \forall\ x\ \in \mathbb{R} - \{0\},$

then $csc\ hy = x, \forall\ y\ \in \mathbb{R} - \{0\}.$

Differentiating: w.r.t "x" both sides, regarding y as a function of x.

$$\therefore \frac{d}{dx}\ (\text{csch}\ y) = \frac{d}{dx}\ (x)$$

$$\Rightarrow \frac{d}{dy}\left(\text{csc}\, h\ y\right) . \frac{dy}{dx} = 1$$

$$\Rightarrow -\text{csc}\, h\, y\, .\coth y\, .\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{csch}\ y\ .\coth y} \qquad (\because \text{csch}\ y > 0 \text{ and } \coth y > 0)$$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc h\, y\sqrt{1+\csc h^2 y-1}}$ $\left(\because \coth y = \sqrt{1+\text{csch}^2 y}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1+x^2}}$

Thus, $\frac{d}{dx}(\text{cosech}^2 x) = \frac{-1}{x\sqrt{1+x^2}}$, $\forall\, x \in \mathbb{R}-\{0\}$.

> **Note:** The derivatives of $\cosh^{-1} x$, $\text{sech}^{-1} x$ and $\coth^{-1} x$ are left as an exercise for readers.

Examples: Find $\frac{dy}{dx}$; when:

(i) $y = \sinh^{-1}(2x+5)$ (ii) $y = \cosh^{-1}(\sec x)$

Solution: (i) Given that

$$y = \sin h^{-1}(2x+5)$$

Differentiating w.r.t "x" by applying chain rule:

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\sin h^{-1}(2x+5)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+(2x+5)^2}} \cdot \frac{d}{dx}(2x+5)$ $\left[\because \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}\right]$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2}{\sqrt{4x^2+20x+26}}}$

Solution: (ii) Given that

$y = \cosh^{-1}(\sec x)$

Differentiating w.r.t. "x" by applying chain rule

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\cosh^{-1}(\sec x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\sec^2 x-1}} \cdot \frac{d}{dx}(\sec x)$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\sec x\,.\,\tan x}{\sqrt{\tan^2 x}} = \sec x}$

List of derivatives of basic standards functions shown in the list 2.

Sr. No	$y = f(x)$	$\frac{dy}{dx}$
1.	$y = \sinh x$	$\cosh x$
2.	$y = \cosh x$	$\sinh x$
3.	$y = \tanh x$	$\text{sech}^2 x$
4.	$y = \coth x$	$-\text{csch}^2 x$
5.	$y = \text{csch}\, x$	$-\text{csch}\, x \,.\coth x$
6.	$y = \text{sech}\, x$	$-\text{sech}\, x \,.\tanh x$
7.	$y = \sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}, -\infty < x < \infty$
8.	$y = \cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, x > 1$
9.	$y = \tanh^{-1} x$	$\frac{1}{1-x^2}, \|x\| < 1$
10.	$y = \coth^{-1} x$	$\frac{1}{1-x^2}, \|x\| > 1$
11.	$y = \text{csch}^{-1} x$	$-\frac{1}{x\sqrt{1+x^2}}, x > 0$
12.	$y = \text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$

3.7.3 Use MAPLE command diff differentiate a function:

- The **diff** command computers the partial derivatives of the expression with respect to $x_1, x_2, \dots x_n$ respectively. The most frequent use is **diff** $(f(x), x)$, which computes the derivative of the function $f(x)$ which respect to x.
- You can enter the **diff** command using either the 1-D or 2-D calling sequence, e.g., **diff** (x, x) is equivalent to $\frac{d}{dx}x$.
- **diff** has a user interface that will call the user's own differentiation functions. If the procedure **"diff"** is defined, then the function call **diff** $(f(x, y, z), y)$ will invoke **diff**/$f(x, y, z)$ to compute the derivative.
- If the derivative cannot be expressed (if the expression is an undefined function), the **diff** function call itself is returned. The pretty printer display the **diff** function in a two-dimensional $\frac{d}{dx}$ format. The differential operator D is also defined in Maple.

Examples. 1 > **diff** $(2x, [x]) = 2$

2. > **diff** $(\sin 2x, [x]) 2\cos 2x$

3. > **diff** $(\sec^2(x), [x]) = 2\sec^2 x . \tan x$

4. > **diff** $(\sqrt{x^3+3}, [x]) \frac{3}{2} \cdot \frac{x^2}{\sqrt{x^3}}$

5. > **diff** $(\cos(\sin(2x)), [x]) = -2\sin(\sin 2x). \cos 2x$

3.7.3 Use MAPLE command diff to differentiate a function

The format of diff command to differentiate a function in MAPLE are as under:

>diff(f, [x]) is equivalent to the command $\frac{d}{dx} f$ in Maple version 2015.

Where,

f stands for function whose derivative is to be evaluated

x stands for the variable x, the derivative with respect x

$\frac{d}{dx}$ means 1st order derivative with respect to variable x

Note: All above operators should be taken from the Maple calculus pallet

Use MAPLE command **diff** or $\left(\frac{d}{dx} f\right)$ to differentiate a function:

Derivative of functions:

$> \frac{d}{dx}(2x^3 + 3x^2 + 5x + 42)$

$$6x^2 + 6x + 5$$

$> \frac{d}{dx} 3\sqrt{x+1}$

$$\frac{1}{3(x+1)^{\frac{2}{3}}}$$

$> \frac{d}{dx} e^{3x}$

$$3e^{3x}$$

$> \frac{d}{dx} \sin(x)$

$$\cos(x)$$

$> \frac{d}{dx} \cos(\sqrt{x})$

$$-\frac{1}{2}\frac{\sin(\sqrt{x})}{\sqrt{x}}$$

$> \frac{d}{dx} \ln(x)$

$$\frac{1}{x}$$

Derivative on Product form:

$> \frac{d}{dx}(e^x\sqrt{x})$

$$e^x\sqrt{x} + \frac{1}{2}\frac{e^x}{\sqrt{x}}$$

$> \frac{d}{dx}(e^x.(x^2+1))$

$$e^x.(x^2+1) + 2(e^x.x)$$

Derivative on Quotient form:

$> \frac{d}{dx}\left(\frac{e^x}{x+3}\right)$

$$\frac{e^x}{x+3} - \frac{e^x}{(x+3)^2}$$

$> \frac{d}{dx}\left(\frac{\ln(x+1)}{\sin(x)}\right)$

$> \frac{d}{dx}(e^x.\sin(x))$

$e^x.\sin(x) + e^x.\cos(x)$

$\frac{1}{(x+1)\sin(x)} - \frac{\ln(x+1)\cos(x)}{\sin^2(x)}$

Exercise 3.6

1. Differentiate the following w.r.t. x:

(i)	$\sinh[\ln(x+3)]$	(ii)	$\sinh(e^{3x})$	(iii)	$\cosh(2x^2+3x)$
(iv)	$\frac{\tanh\sqrt{x}}{\sqrt{\cosh x}}$	(v)	$\tan(e^{\sinh^{-1}x})$	(vi)	$\frac{\sinh^{-1}x}{\text{sech}^{-1}x}$
(vii)	$\cosh x\,.\coth x^2$	(viii)	$\sinh x \tanh x^2$	(ix)	$\ln[\tanh(x^2+2x+1)]$

2. Find $\frac{dy}{dx}$, for the following functions:

(i)	$y = x\cosh^{-1}x - \sqrt{x^2-1}$	(ii)	$y = x\tanh^{-1}(3x)$
(iii)	$\ln(\cosh^{-1}x) + \sinh^{-1}y = C$	(iv)	$y = \ln(1-x^2) + 2x\tanh^{-1}x$
(v)	$y = \tanh^{-1}(\tan x^3)$	(vi)	$y = x\,\text{sech}^{-1}(\sqrt{x})$

3. Write MAPLE command **diff** to differentiate the following:

(i)	$f(x) = 2x^3 + 3x^2 + 6$	(ii)	$f(x) = \sin(2x+3)$
(iii)	$f(x) = (x+1)(x+2)$	(iv)	$f(x) = \frac{x^2-3x+2}{x^2-4}$

4. Write MAPLE command $> \frac{d}{dx}$ to differentiate the following functions:

(i)	$f(x) = x^3 + 5x^2 + 3x + 7$	(ii)	$f(x) = \sin x^2$
(iii)	$f(x) = \frac{\sqrt{x}+1}{x^2+1}$		

Review Exercise 3

1. Select the correct options:

(i) The derivative of $\frac{2}{x^3}$ is:

(a) $\frac{2}{3x^2}$ (b) $-\frac{2}{3x^2}$ (c) $\frac{6}{x^4}$ (d) $-\frac{6}{x^4}$

(ii) The derivative of $\sqrt{x} + x\sqrt{x}$ is:

(a) $\frac{1}{2\sqrt{x}} + \frac{3\sqrt{x}}{2}$ (b) $-\frac{1+x}{2\sqrt{x}}$ (c) $\frac{1}{\sqrt{x}} + 3\sqrt{x}$ (d) $\frac{1}{\sqrt{x}} + \frac{2\sqrt{x}}{3}$

(iii) If $y = (x+1)(x^2-2)$, then $\frac{dy}{dx}$ is:

(a) $x^3 + x^2 - 2x - 2$ (b) $3x^2 + 2x - 2$ (c) $3x^2 - 2x + 2$ (d) $3x^2 + 2x + 2$

(iv) If $ax^2 + by^2 = ab$, then $\frac{dy}{dx}$ is:

(a) $\frac{-2ax}{by}$ (b) $\frac{-bx}{ay}$ (c) $\frac{-ax}{by}$ (d) $\frac{-ax}{2by}$

(v) If $y = \sqrt{\tan x - y}$, then $\frac{dy}{dx} = ?$

(a) $\frac{\tan x}{2y+1}$ (b) $\frac{-\csc^2 x}{2y-1}$ (c) $\frac{\sec^2 x}{2y-1}$ (d) $\frac{\sec^2 x}{2y+1}$

(vi) If $y = \tan^{-1}\sqrt{x}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{x+\sqrt{x}}$ (c) $\frac{1}{2(x+x\sqrt{x})}$ (d) $\frac{1}{2(\sqrt{x}+x\sqrt{x})}$

(vii) The derivative of $\tan x$ w.r.t. $\cot x$ is:

(a) $\sec^2 x \csc^2 x$ (b) $-\tan^2 x$ (c) $\frac{\sec^2 x}{\csc^2 x}$ (d) $\tan^2 x$

(viii) The $f(x) = ax^2 - 3x - 5$ and $f'(2) = 9$, then a is equal to:

(a) -2 (b) 3 (c) 4 (d) 5

(ix) The derivative of x^2e^{2x} is:

(a) $x^2e^{2x} + 2x^2e^x$ (b) $2xe^{2x}$ (c) $2e^{2x}(x^2+x)$ (d) $2e^{2x}(x^2+1)$

(x) The derivative of a^x, if $a < 0$ is:

(a) $-a^x \ln a$ (b) $a^x.\ln a$ (c) $\frac{a^x}{\ln a}$ (d) Does not exist

(xi) If $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ then $\frac{dy}{dx} = ?$

(a) 1 (b) -1 (c) 2 (d) $\frac{1}{2}$

(xii) $\frac{d}{dx}(\sinh^{-1} x + \cosh^{-1} x)$ is:

(a) $\cosh^{-1} x - \sinh^{-1} x$ (b) $\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2-1}}$ (d) $\frac{1}{\sqrt{x^2-1}} - \frac{1}{\sqrt{1+x^2}}$

(xiii) The derivative of $\tanh ax$ is:

(a) $\text{sech}^2 ax$ (b) $a\,\text{sech}\, ax$ (c) $a\,\text{sech}^2 ax$ (d) $2a\,\text{sech}^2 ax$

(xiv) The derivative of $\coth^{-1}(2x)$ is:

(a) $\frac{1}{1-4x^2}$ (b) $\frac{2}{1-4x^2}$ (c) $\frac{2x}{1-4x^2}$ (d) $\frac{2}{1-x^2}$

(xv) f is the function with rule $f(x) = \ln 2x$ $(x > 0)$, if g is the inverse of f, then $g'(x) =$

(a) $\frac{2}{x}$ (b) $\frac{1}{2x}$ (c) $\frac{2}{e^x}$ (d) $\frac{e^x}{2}$

(xvi) If $f(x) = a\cos 3x$ and $f'\left(\frac{\pi}{2}\right) = 6$, then $a =$

(a) -6 (b) -2 (c) 2 (d) 3

2. Find the derivative of $\sqrt{\cos x}$ and $\sec\sqrt{x}$ by first principle.

3. If $y = (\sin x)^{\ln x}$, find $\frac{dy}{dx}$.

4. Find $\frac{dy}{dx}$, if $ax^2 + 2hxy + by^2 = 0$.

5. Let $f(x) = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$, find $f'(x)$ and $f'(-\sqrt{3})$.

6. $x = 4(t - \sin t)$ and $y = 4(1 + \cos t)$, find $\frac{dy}{dx}$.

7. Differentiate w.r.t. x:

(i) $\frac{x^2+x^{-1}}{x^2-x^{-1}}$ (ii) $\frac{3x-2}{\sqrt{x^2+1}}$

8. If $y = x^4 + 2x^2$, show that $\frac{dy}{dx} = 4x\sqrt{y+1}$.

9. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots}}}$, show that $(2y-1)y' = \cos x$.

10. Differentiate w.r.t. x:

(i) $\cosh(\cos^{-1}\sqrt{x})$ (ii) $\tanh^{-1}(\cos 2x)$.

Unit 4 Higher Order Derivatives and Applications

4.1 Higher Order Derivatives

The derivative of a function $y = f(x)$ is $\frac{dy}{dx} = f'(x)$, which is itself a function. Now the derivative of $\frac{dy}{dx} = f'(x)$, written as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x),$$

Generally it is referred as the second order derivative of $f(x)$ and this differentiation process can be continued to find the third, fourth,…., nth order derivative as under, and are called **higher order derivatives** of $f(x)$.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y''' = f'''(x)$$

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4} = y^4 = f^4(x)$$

... .

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$$

4.1.1 Find higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions.

(i) Higher order derivatives of algebraic functions:

Example 1. Find the first, second, and third order derivatives of

$$y = 5x^4 - 3x^3 + 7x^2 - 9x + 2$$

Solution: We have $y = 5x^4 - 3x^3 + 7x^2 - 9x + 2$

Differentiating w.r.t x, we get

$$\Rightarrow \quad \frac{dy}{dx} = 20x^3 - 9x^2 + 14x - 9$$

Again, differentiating w.r.t x, we get

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 60x^2 - 18x + 14$$

Differentiating 3rd time w.r.t x, we get

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 120x - 18$$

Example 2. Find $f'''(4)$ if $y = f(x) = \sqrt{x}$

Solution: As $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Differentiating successively three times, we have

$$\Rightarrow \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\Rightarrow \quad f''(x) = \frac{-1}{4}x^{-\frac{3}{2}}$$

$$\Rightarrow \quad f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

Replacing x by 4, we get

Hence, $$f'''(4) = \frac{3}{8}(4)^{-\frac{5}{2}}$$

$$= \frac{3}{8}\left(\frac{1}{32}\right) = \frac{3}{256}$$

Example 3. If $f(x) = \frac{2}{1-x}$ then find $f^{(n)}(x)$.

Solution:

$$f(x) = \frac{2}{1-x} = 2(1-x)^{-1}$$

Differentiating successively w.r.t x and patterning for nth derivative, we have

$$f'(x) = 2(-1)(1-x)^{-2}(-1) = 2(1!)(1-x)^{-2}$$

$$f''(x) = 2(1!)(-2)(1-x)^{-3}(-1) = 2(2!)(1-x)^{-3}$$

$$f'''(x) = 2(2!)(-3)(1-x)^{-4}(-1) = 2(3!)(1-x)^{-4}$$

$$f^{(4)}(x) = 2(3!)(-4)(1-x)^{-5}(-1) = 2(4!)(1-x)^{-5}$$

Therefore,

$$f^{(n)}(x) = 2(n!)(1-x)^{-(n+1)}$$

(ii) Higher order derivatives of trigonometric function:

The higher order derivatives of trigonometric functions are explained in the following examples.

Example 1. Find the third order derivative of $y = sin^2x$.

Solution: As $y = sin^2x$

Differentiating successively thrice times w.r.t x, we have

$$\Rightarrow \quad \frac{dy}{dx} = 2\sin x\cos x = \sin 2x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos 2x$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 2(-\sin 2x)(2)$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = -4\sin 2x$$

Example 2. Find 2nd order derivative of $f(x) = \frac{\cos x}{1+\sin x}$

Solution: $f(x) = \frac{\cos x}{1+\sin x}$

Differentiating successively two times w.r.t x, we have

$$\Rightarrow \quad f'(x) = \frac{(1+\sin x)(-\sin x) - \cos x\ \cos x}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{-1}{1+\sin x}$$

$$\Rightarrow \quad f''(x) = -\frac{d}{dx}(1+\sin x)^{-1} = (1+\sin x)^{-2}\cos x$$

$$\Rightarrow \quad f''(x) = \frac{\cos x}{(1+\sin x)^2}$$

(iii) Higher order derivatives of exponential function:

Example 1. Find the 3rd order derivative of $y = a^x$

Solution: As $y = a^x$

$y = e^{xlna}$ $\qquad [\because a^x = e^{xlna}]$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \quad y' = e^{xlna}.(lna) = lna\, e^{xlna}$$

$$\Rightarrow \quad y'' = (lna)\, e^{xlna}.(lna) = (lna)^2\, e^{xlna}$$

$$\Rightarrow \quad y''' = (lna)^2\, e^{xlna}.(lna) = (lna)^3\, e^{xlna}$$

or $\quad y''' = a^x(\ln a)^3$

Example 2. Find the 2nd order derivative of $f(x) = e^{1+x^2}$

Solution: $f(x) = e^{(1+x^2)}$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \quad f'(x) = e^{(1+x^2)} \cdot 2x$$

$$\Rightarrow \quad f''(x) = 2\left[xe^{(1+x^2)} \cdot 2x + e^{(1+x^2)} \cdot 1\right]$$

$\Rightarrow \qquad = 2\left[2x^2e^{(1+x^2)} + e^{(1+x^2)}\right]$

$\Rightarrow \qquad f''(x) = 2e^{(1+x^2)}(2x^2+1)$

(iv) Higher order derivatives of logarithmic function:

The higher order derivatives of logarithmic functions are explained in the following examples.

Example 1. Find 3rd order derivative of $f(x) = \log_b x^2$

As $f(x) = \log_b x^2$

Solution: $y = \log_b x^2$

or $\quad = 2\log_b x$

Differentiating successively three times w.r.t x, we have

$$\Rightarrow \quad y' = \frac{2}{x}\cdot\frac{1}{\ln b} \qquad \left[\because \frac{d}{dx}[log_b x] = \frac{1}{x}\cdot\frac{1}{\ln b}\right]$$

$$\Rightarrow \quad y'' = \frac{2}{\ln b}\cdot\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad y''' = \frac{2}{\ln b}\cdot\left(\frac{2}{x^3}\right) = \frac{4}{x^3\ln b}$$

Example 2. Find the 2nd order derivative of $f(x) = \ln(1+x^2)$

Solution: $\qquad f(x) = \ln(1+x^2)$

Differentiating successively two times w.r.t x, we have

$$\Rightarrow \quad f'(x) = \frac{1}{(1+x^2)}\frac{d}{dx}(1+x^2) = \frac{1}{(1+x^2)}.2x$$

$$\Rightarrow \quad f''(x) = 2\frac{d}{dx}\left(\frac{x}{1+x^2}\right)$$

$$= 2\left[\frac{(1+x^2)(1) - x\,(2x)}{(1+x^2)^2}\right] = 2\left[\frac{1+x^2-2x^2}{(1+x^2)^2}\right]$$

$$\Rightarrow \quad f''(x) = 2\left[\frac{1-x^2}{(1+x^2)^2}\right]$$

4.1.2 Find the second derivative of implicit, inverse trigonometric and parametric functions

(i) 2nd order derivatives of implicit function:

The method of finding the second order derivatives of implicit functions is explained in the following examples.

Example: Find $\dfrac{d^2y}{dx^2}$ if $xy + x - 2y - 1 = 0$

Solution: As $xy + x - 2y - 1 = 0$

Differentiating w.r.t x, we have

$$\frac{d}{dx}(xy)+\frac{d}{dx}(x)-2\frac{d}{dx}(y)-\frac{d}{dx}(1)=\frac{d}{dx}(0)$$

$$\Rightarrow \quad xy'+y\cdot 1+1-2y'-0=0$$

$$\Rightarrow \quad y'(x-2)=-(y+1)$$

$$\Rightarrow \quad y'=\frac{-(y+1)}{(x-2)}$$

Differentiating y', to get 2nd order derivative, we have

$$\Rightarrow \quad y''=-\frac{(x-2)\frac{d}{dx}(y+1)-(y+1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$\Rightarrow \quad y''=-\frac{(x-2)y'-(y+1)(1)}{(x-2)^2}$$

$$\Rightarrow \quad y''=-\frac{(x-2)\left[-\frac{(y+1)}{(x-2)}\right]-(y+1)}{(x-2)^2}$$

$$\Rightarrow \quad y''=\frac{2(y+1)}{(x-2)^2}$$

(ii) 2nd order derivatives of inverse trigonometric function:

The method of finding second order derivatives of inverse trigonometric functions is explained in the following examples.

Example: Find the second order derivative of $\tan^{-1}x$

Solution:

Let $y=\tan^{-1}x$

Differentiating y w.r.t x, we get

$$\frac{d}{dx}(y)=\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+x^2}=(1+x^2)^{-1}$$

Again, differentiating w.r.t x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(1+x^2)^{-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-1(1+x^2)^{-2}(2x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{-2x}{(1+x^2)^2}$$

(iii) 2nd order derivatives of parametric function:

Let $y = f(x)$ is a function, $x = f(t)$ and $y = g(t)$ are the parametric equations of $y = f(x)$. Then, by using chain rule.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \ldots(i)$$

To find $\frac{d^2y}{dx^2}$, let $z = \frac{dy}{dx} = h(t)$.

Now, using chain rule

$$\frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}}$$

From equation (i), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} \qquad \because\ z = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

Example: Find $\frac{d^2y}{dx^2}$ where $y = 1 + 5t^2$; $x = 5t + 3t^2$ are parametric equation of $y = f(x)$.

Solution:

By using formula $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

Here $\frac{dy}{dt} = \frac{d}{dt}(1 + 5t^2) = 10t$

$\frac{dx}{dt} = \frac{d}{dt}(5t + 3t^2) = 5 + 6t$

Now $y' = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{10t}{5+6t}$

For second order derivative we use the following formula

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{10t}{5+6t}\right)$$
$$= \frac{(5+6t)(10)-(10t)(6)}{(5+6t)^2} = \frac{(50+60t-60t)}{(5+6t)^2} = \frac{50}{(5+6t)^2}$$

By using formula $y'' = \frac{d^2y}{dx^2} = \frac{\frac{50}{(5+6t)^2}}{5+6t}$

$$y'' = \frac{50}{(5+6t)^3}$$

4.1.3 Use MAPLE command diff repeatedly to find higher order derivative of a function

The format of diff command to differentiate a function repeatedly in MAPLE is as under:

>diff (f ^{n}, [x]) is equivalent to the command $\frac{d^n}{dx^n} f$ in Maple version 2022.

Where,

f^n stands for function whose nth order derivative is to be evaluated

x stands for the variable x, for the required derivative with respect x.

$\frac{d^n}{dx^n}$ means n[th] order derivative with respect to variable x.

Note: All above operators should be taken from the Maple calculus pallet

Use MAPLE command **diff** repeatedly or $\frac{d^n}{dx^n} f$ to differentiate a function repeatedly:

***n*th order Derivative of functions:**

$> \frac{dy}{dx}(x^3+2x^2+5x+7)$ $3x^2+4x+5$ $> \frac{d^2}{dx^2}(x^3+2x^2+5x+7)$ $6x+4$ $> \frac{d^3}{dx^3}(x^3+2x^2+5x+7)$ 6	$> \frac{d}{dx}\sqrt{x^2+4x+3}$ $\frac{1}{2}\frac{2x+4}{\sqrt{x^2+4x+3}}$ $> \frac{d^2}{dx^2}\left(\sqrt{x^2+4x+3}\right)$ $-\frac{1}{4}\frac{(2x+4)^2}{(x^2+4x+3)^{\frac{3}{2}}}+\frac{1}{\sqrt{x^2+4x+3}}$ $> \frac{d^3}{dx^3}\left(\sqrt{x^2+4x+3}\right)$ $\frac{3}{8}\frac{(2x+4)^3}{(x^2+4x+3)^{\frac{5}{2}}}-\frac{3}{2}\frac{2x+4}{(x^2+4x+3)^{\frac{3}{2}}}$

> $\frac{d}{dx}\sin(x)$ $\cos(x)$ > $\frac{d^2}{dx^2}\sin(x)$ $-\sin(x)$ > $\frac{d^3}{dx^3}\sin(x)$ $-\cos(x)$	> $\frac{d^3}{dx^3}(e^{2x+5})$ $8e^{2x+5}$ > $\frac{d^3}{dx^3}(\ln x^2)^3$ $120\ln^3 x^3$
Derivative on Product form:	**Derivative on Quotient form:**
> $\frac{d^2}{dx^2}(e^{2x+1})(\sqrt{x})$ $\frac{1}{4}\frac{D^{(2)}(e^{2x+1})(\sqrt{x})}{x}-\frac{1}{4}\frac{D(e^{2x+1})(\sqrt{x})}{x^{\frac{3}{2}}}$ > $\frac{d}{dx}(e^x.(x^2+1))$ $e^x.(x^2+1)+2(e^x.x)$ > $\frac{d^3}{dx^3}(e^x\sin(x))$ $2e^x\cos(x)-2e^x\sin(x)$	> $\frac{d^3}{dx^3}\left(\frac{e^x}{x+3}\right)$ $\frac{e^x}{x+3}-\frac{3e^x}{(x+3)^2}+\frac{6e^x}{(x+3)^3}-\frac{6e^x}{(x+3)^4}$ > $\frac{d^3}{dx^3}\left(\frac{\ln(x+1)}{\sin x}\right)$ $\frac{2}{(x+1)^3\sin x}$ > $\frac{d^3}{dx^3}\left(\frac{1+\sin x}{\cos x}\right)$ $\frac{6\sin}{\cos x^3}-\frac{6(\sin x+1)}{\cos x^4}$

Exercise 4.1

1. Calculate the first, second and third order derivatives of $y=\cos^2 x$.
2. Find the 2nd order derivative of $f(x)=\frac{\cos x}{1+\sin x}$.
3. Find the fourth order derivatives of the given functions.
   (i) $h(t)=3t^7-6t^4+8t^3-12t+18$
   (ii) $f(x)=\sqrt[3]{x}-\frac{1}{8x^2}-\sqrt{x}$
4. Determine the fourth order derivative in each of the following function.
   (i) $r(t)=3t^2+8\sqrt{t}$ (ii) $y=\cos x$ (iii) $f(y)=\sin 3y+e^{-2y}+\ln(7y)$
5. If $x^2+y^2=10$, find y''.
6. Find $\frac{d^2y}{dx^2}$ if
   (i) $2y^2+6x^2=76$ (ii) $x^3+y^3=1$

7. Find $\frac{d^2y}{dx^2}$ if

   (i) $x = -5t^3 - 7$ and $y = 3t^2 + 16$

   (ii) $x = \cos\theta$ and $y = \sin\theta$

8. The derivative of function $r(t)$ is given by

   $r'(t) = 6t + 4t^{-1/2} + e^t$; find $r''(t), r'''(t)$ and $r^{(4)}(t)$

9. If $x^2 + y^2 = 25$ then find $\frac{d^2y}{dx^2}$ at point (4,3).

12. Write MAPLE Command to find higher order derivatives of the following functions:

   (i) $f(x) = x^3 + 3x^2 + 6x + 8$; (third order derivative)

   (ii) $f(x) = \cos\sqrt{2x+3}$; (second order derivative)

   (iii) $f(x) = e^{(x^2+5x+3)}$; (third order derivative)

   (iv) $f(x) = \ln\sqrt{3x+2}$; (second order derivative)

   (v) $f(x) = e^{\sin x}$; (third order derivative)

4.2 Maclaurin's and Taylor's Expansions

4.2.1 State Maclaurin's and Taylor's theorems (without remainder terms). Use these theorems to expand $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $log_a(1+x)$ and $\ln(1+x)$

Maclaurin's Theorem:

If $f(x)$ is nth order differentiable function at $x = 0$ then it can be expanded as the infinite sum of the terms of the polynomial centered at $x = 0$ that is

i.e., $$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

Taylor's Theorem:

If $f(x)$ is nth order differentiable function at $x = a$ then it can be expanded as the infinite sum of the terms of the polynomial centered at $x = a$

i.e., $$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

Example: Find the Maclaurin's series of $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $log_a(1+x)$ and $ln(1+x)$.

Solution:

(i) $f(x) = \sin x$

The Maclaurin's series is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

$$
\begin{aligned}
&f(x) = \sin x && \text{at} && \Rightarrow f(0) = 0\\
&f'(x) = \cos x && && \Rightarrow f'(0) = 1\\
&f''(x) = -\sin x && && \Rightarrow f''(0) = 0\\
&f'''(x) = -\cos x && && \Rightarrow f'''(0) = -1\\
&f^{(4)}(x) = \sin x && && \Rightarrow f^{(4)}(0) = 0\\
&f^{(5)}(x) = \cos x && && \Rightarrow f^{(5)}(0) = 1
\end{aligned}
$$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0), \ldots,$ in Maclaurin's series we get,

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) - \cdots$$

or $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

In summation form,

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This is the required Maclaurin's series of the function $f(x) = \sin x$.

(ii) $\boldsymbol{f(x) = \tan x}$

$$
\begin{aligned}
&f(x) = \tan x = y && \text{at} \quad \Rightarrow f(0) = 0\\
&f'(x) = y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 && \Rightarrow f'(0) = 1\\
&f''(x) = y'' = 2yy' && \Rightarrow f''(0) = 0\\
&f'''(x) = y''' = 2[yy'' + y'^2] && \Rightarrow f'''(0) = 2\\
&f^{(4)}(x) = y^{\text{iv}} = 2[yy''' + y''y'] + 2.2y'y'' && \Rightarrow f^{(4)}(0) = 0\\
&\qquad = 2[yy''' + y''.y' + 2y'y'']\\
&\qquad = 2[yy''' + 3y'y'']\\
&f^{(5)}(x) = y^5 = 2[yy^{(4)} + y'''y'] + 6[y'y''' + y''y'']\\
&\qquad = 2yy^{(4)} + 2y'''y' + 6y'y''' + 6y''y''\\
&\qquad = 2yy^{(4)} + 8y'y''' + 6y''y''\\
&\qquad = 2yy^{(4)} + 8y'(2) + 6y''y''\\
&\qquad = 2yy^{(4)} + 16y' + 6y''y'' && \Rightarrow f^5(0) = 16
\end{aligned}
$$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$ and $f^{(5)}(0)$ in Maclaurin's series, we get,

$$\tan x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

This is the required Maclaurin's series of the function $f(x) = \tan x$

(iii) $\boldsymbol{f(x) = a^x}$

$f(x) = a^x$ at $x = 0$ $\Rightarrow f(0) = 1$

$f'(x) = a^x \ln a$ $\Rightarrow f'(0) = \ln a$

$f''(x) = \ln a\,(a^x \ln a) = (\ln a)^2 a^x$ $\Rightarrow f''(0) = (\ln a)^2$

$f'''(x) = (\ln a)^2 (a^x \ln a) = (\ln a)^3 a^x$ $\Rightarrow f'''(0) = (\ln a)^3$

$f^{(4)}(x) = (\ln a)^3 (a^x \ln a) = (\ln a)^4 a^x$ $\Rightarrow f^{(4)}(0) = (\ln a)^4$

$f^{(5)}(x) = (\ln a)^4 (a^x \ln a) = (\ln a)^5 a^x$ $\Rightarrow f^{(5)}(0) = (\ln a)^5$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$ and $f^{(5)}(0)$ in Maclaurin's series, we get

$$a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \frac{x^4}{4!}(\ln a)^4 + \frac{x^5}{5!}(\ln a)^5 + \cdots$$

$$\Rightarrow \quad a^x = 1 + (x \ln a) + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \frac{(x \ln a)^4}{4!} + \frac{(x \ln a)^5}{5!} + \cdots$$

In summation form,

$$a^x = \sum_{n=0}^{\infty} \frac{(x \ln a)^n}{n!}$$

This is the required Maclaurin's series of the function $f(x) = a^x$

(iv) $\boldsymbol{f(x) = \log_a(1 + x)}$**, where** $\boldsymbol{a > 0}$ **and** $\boldsymbol{a \neq 1}$

$f(x) = \log_a(1 + x)$ $\Rightarrow$ $f(0) = 0$

$f'(x) = \frac{1}{(1+x)\ln a}$ $\Rightarrow$ $f'(0) = \frac{1}{\ln a}$

or $f'(x) = \frac{(1+x)^{-1}}{\ln a}$

$f''(x) = -\frac{(1+x)^{-2}}{\ln a}$ $\Rightarrow$ $f''(0) = -\frac{1}{\ln a}$

$f'''(x) = \frac{2(1+x)^{-3}}{\ln a}$ $\Rightarrow$ $f'''(0) = \frac{2}{\ln a}$

$f^{(4)}(x) = \frac{-6(1+x)^{-4}}{\ln a}$ $\Rightarrow$ $f^{(4)}(0) = \frac{-6}{\ln a}$

$f^{(5)}(x) = \frac{24(1+x)^{-5}}{\ln a}$ $\Rightarrow$ $f^{(5)}(0) = \frac{24}{\ln a}$

Putting values of $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0), f^{(5)}(0)$ in Maclaurin's series, we get

$$\log_a(1 + x) = 0 + x\left(\frac{1}{\ln a}\right) + \frac{x^2}{2!}\left(\frac{-1}{\ln a}\right) + \frac{x^3}{3!}\left(\frac{2}{\ln a}\right) + \frac{x^4}{4!}\left(\frac{-6}{\ln a}\right) + \frac{x^5}{5!}\left(\frac{24}{\ln a}\right) + \cdots$$

$$\Rightarrow \quad \log_a(1+x) = \frac{1}{\ln a}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right]$$

In summation form,

$$\log_a(1+x) = \frac{1}{\ln a}\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{n+1}}{n+1}$$

Which is the required Maclaurin's series.

Note: Maclurin's series of $\cos x\,, e^x$ and $\ln(1+x)$ are left as an exercise for readers.

Find the Taylor's series of the expansion of $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $\log_a(1+x)$ and $\ln(1+x)$ at particular point a

Solution: (i) $f(x) = \sin x$ **at** $a = \frac{\pi}{6}$

The Taylor's series is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

At $a = \frac{\pi}{6}$; $f(x) = f\left(\frac{\pi}{6}\right) + \left(x - \frac{\pi}{6}\right)f'\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2!}f''\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{3!}f'''\left(\frac{\pi}{6}\right) + \cdots$ ...(i)

$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \quad \Rightarrow \quad f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$f'''(x) = -\cos x \quad \Rightarrow \quad f'''\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

Putting these above values in equation (i), we get

$$f(x) = \frac{1}{2} + \left(x - \frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{\left(x - \frac{\pi}{6}\right)^2}{2!}\left(-\frac{1}{2}\right) + \frac{\left(x - \frac{\pi}{6}\right)^3}{3!}\left(-\frac{\sqrt{3}}{2}\right) + \cdots$$

$$\Rightarrow \sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\frac{\left(x - \frac{\pi}{6}\right)^2}{2!} - \frac{\sqrt{3}}{2}\frac{\left(x - \frac{\pi}{6}\right)^3}{3!} + \cdots$$

Which is the required Taylor series of $\sin x$ at the point $\frac{\pi}{6}$.

(ii) $f(x) = \cos x$ **at** $a = \frac{\pi}{4}$

The Taylor's series at $a = \frac{\pi}{4}$ is given by:

$$f(x) = f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \cdots \quad \ldots\text{(i)}$$

$$f(x) = \cos x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(x) = -\sin x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f''(x) = -\cos x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = \sin x \quad \Rightarrow \quad f'''\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

By putting values in equation (i), we get

$$f(x) = \frac{1}{\sqrt{2}} + \left(x - \frac{\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\frac{\left(x - \frac{\pi}{4}\right)^2}{2!} - \frac{1}{\sqrt{2}}\frac{\left(x - \frac{\pi}{4}\right)^3}{3!} + \cdots$$

Which is the required Taylor series of $\cos x$ at the point $\frac{\pi}{4}$.

(iii) $\boldsymbol{f(x) = \tan x}$ **at** $\boldsymbol{a = \frac{\pi}{4}}$

$f(x) = \tan x = y$ at $x = \frac{\pi}{4} \Rightarrow f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \quad \Rightarrow f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = y'' = 2yy' \quad \Rightarrow f''\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = y''' = 2[yy'' + y'^2] \quad \Rightarrow f'''\left(\frac{\pi}{4}\right) = 16$

$$\begin{aligned} f^4(x) = y^4 &= 2[yy''' + y''y' + 2y'y''] \\ &= 2[yy''' + y''.y' + 2y'y''] \\ &= 2[yy''' + 3y'y''] \qquad \Rightarrow f^4\left(\frac{\pi}{4}\right) = 80 \end{aligned}$$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^4(0), \ldots,$ in Taylor series we get,

$$f(x) = 1 + \frac{2}{1!}\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{80}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \cdots$$

This is the required Taylor series of the function $f(x) = \tan x$ at $a = \frac{\pi}{4}$

(iv) $\boldsymbol{f(x) = a^x}$ **at** $\boldsymbol{b = 2}$

$f(x) = a^x$ at $x = 2 \Rightarrow f(2) = a^2$

$f'(x) = a^x \ln a$ $\Rightarrow f'(2) = a^2 \ln a$

$f''(x) = \ln a\,(a^x \ln a) = (\ln a)^2 a^x$ $\Rightarrow f''(2) = a^2(\ln a)^2$

$f'''(x) = (\ln a)^2(a^x \ln a) = (\ln a)^3 a^x$ $\Rightarrow f'''(2) = a^2(\ln a)^3$

The Taylor's series of the function at point b is given by

$$f(x) = f(b) + (x-b)f'(b) + \frac{(x-b)^2}{2!}f''(b) + \frac{(x-b)^3}{3!}f'''(b) + \cdots$$

At $b = 2$, $f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$

Putting values of $f(2), f'(2), f''(2), f'''(2), \ldots$ we get

$$f(x) = a^2 + (x-2)(a^2 \ln a) + \frac{(x-2)^2}{2!}(a^2(\ln a)^2) + \frac{(x-2)^3}{3!}(a^2(\ln a)^3) + \cdots$$

$$\Longrightarrow \quad a^x = a^2\left[1 + (x-2)\ln a + \frac{(x-2)^2}{2!}(\ln a)^2 + \frac{(x-2)^3}{3!}(\ln a)^3 + \cdots\right]$$

Which is the required Taylor's series of a^x at the point 2.

(v) $\boldsymbol{f(x) = e^x}$ **at** $\boldsymbol{a = 1}$

$f(x) = e^x$ at $x = 1 \Rightarrow f(1) = e$

$f'(x) = e^x.\ln e = e^x$ $\Rightarrow f'(1) = e$

$f''(x) = e^x$ $\Rightarrow f''(1) = e$

$f'''(x) = e^x$ $\Rightarrow f'''(1) = e$

By putting the values of $f(0), f'(0), f''(0), f'''(0), \ldots,$ in Taylor series we get,

$$\Longrightarrow \quad e^x = e + (x-1)(e) + \frac{(x-1)^2}{2!}(e) + \frac{(x-1)^3}{3!}(e) + \cdots$$

$$\Longrightarrow \quad e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

This is the required Taylor series of the function $f(x) = e^x$ at $a = 1$

(vi) $\boldsymbol{f(x) = \log_a(1+x)}$ **at** $\boldsymbol{b = 1}$

Solution: By Taylor's series we have,

$$f(x) = f(b) + (x-b)f'(b) + \frac{(x-b)^2}{2!}f''(b) + \frac{(x-b)^3}{3!}f'''(b) + \cdots$$

At $b = 1$, we have

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots \qquad \ldots(i)$$

$\therefore$ $f(x) = \log_a(1+x)$ $\Rightarrow$ at $\Rightarrow f(1) = \log_a 2$

$f'(x) = \frac{1}{(1+x)\ln a}$ $\Rightarrow$ $f'(1) = \frac{1}{2\ln a}$

or $f'(x) = \frac{(1+x)^{-1}}{\ln a}$

$f''(x) = \frac{-(1+x)^{-2}}{\ln a}$ $\Rightarrow$ $f''(1) = \frac{-1}{4\ln a}$

$f'''(x) = \frac{2(1+x)^{-3}}{\ln a}$ $\Rightarrow$ $f'''(1) = \frac{1}{4\ln a}$

Putting values in equation (i), we get

$$f(x) = \log_a 2 + (x-1) \times \frac{1}{2\ln a} + \frac{(x-1)^2}{2!} \times \frac{(-1)}{4\ln a} + \frac{(x-1)^3}{3!} \times \frac{1}{4\ln a} + \cdots$$

$$\Rightarrow f(x) = \log_a 2 + \frac{(x-1)}{2\ln a} - \frac{(x-1)^2}{4.2!\,(\ln a)} + \frac{(x-1)^3}{4.3!\,(\ln a)} + \cdots$$

Which is the required Taylor's series of the function

$f(x) = \log_a(1+x)$ at $b = 1$

(vii) $\boldsymbol{f(x) = \ln(1+x)}$ **at** $\boldsymbol{b = 2}$

$f(x) = \ln(1+x)$ at $\Rightarrow f(2) = \ln 3$

$f'(x) = \frac{1}{(1+x)} = (1+x)^{-1}$ $\Rightarrow f'(2) = \frac{1}{3}$

$f''(x) = -(1+x)^{-2}$ $\Rightarrow f''(2) = -\frac{1}{9}$

$f'''(x) = 2(1+x)^{-3}$ $\Rightarrow f'''(2) = \frac{2}{27}$

$f^{iv}(x) = -6(1+x)^{-4}$ $\Rightarrow f^{iv}(2) = -\frac{2}{27}$

By putting the values of $f(0), f'(0), f''(0), f'''(0), f^{iv}(0), \ldots,$ in Taylor series we get,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$$

$$\ln(1+x) = \ln 3 + (x-2)\left(\frac{1}{3}\right) + \frac{(x-2)^2}{2!}\left(-\frac{1}{9}\right) + \frac{(x-2)^3}{3!}\left(\frac{2}{27}\right) + \frac{(x-2)^4}{4!}\left(-\frac{2}{27}\right) + \cdots$$

$$\ln(1+x) = \ln 3 + \frac{1}{3}(x-2) - \frac{1}{18}(x-2)^2 + \frac{1}{81}(x-2)^3 - \frac{1}{324}(x-2)^4 + \cdots$$

This is the required Taylor series of the function $f(x) = \ln(1+x)$ at point 2.

4.2.2 Use MAPLE command Taylor to find Taylor's expansion for a given function

The format of Taylor's expansion command in MAPLE is as under:

> taylor (f(x), x = a,n)

where,

f(x) is the function whose Taylor's expansion is required x=a is about the point x=a, series is expanded

n is the number of terms series expanded.

In order to compute the Taylor series expansion following examples are given:

$> taylor\left(\frac{1}{\sqrt{1+x}}, x=0,5\right)$

$1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4+0(x^5)$

$> taylor\,(e^x, x=0,5)$

$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+0(x^5)$

$> taylor\,(e^x, x=2,5)$

$e^2+e^2(x-2)+\frac{1}{2}e^2(x-2)^2+\frac{1}{6}e^2(x-2)^3+\frac{1}{24}e^2(x-2)^4+0((x-2)^5)$

$> taylor\left(\frac{1}{\sqrt{1+x}}, x=0,6\right)$

$1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4-\frac{35}{128}x^5+0(x^6)$

$> taylor\,(\sin(x), x=0,10)$

$x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\frac{1}{362880}x^9+0(x^{11})$

$> taylor(\ln(1+x), x=1,4)$

$\ln(2)+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{24}(x-1)^3+0((x-1)^4)$

$> taylor\,(\cos(x), x=0,10)$

$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8+0(x^{10})$

$> taylor(ln(1+x), x=2,4)$

$\ln(3)+\frac{1}{3}(x-2)-\frac{1}{18}(x-2)^2+\frac{1}{81}(x-2)^3+0((x-2)^4)$

Exercise 4.2

1. Obtain the first three terms of the Maclaurin's series for

   (i) $\cos x$ (ii) e^x (iii) $\ln(1+x)$ (iv) $\sin^2 x$

   (v) $e^{\sin x}$ (vi) xe^{-x} (vii) $\frac{1}{1+x}$

2. Find the first four terms of the Taylor's series for the following functions

   (i) $\ln x$ centered at $a=1$ (ii) $\frac{1}{x}$ centered at $a=1$

   (iii) $sin\, x$ centered at $a=\frac{\pi}{4}$ (iv) $\cos x$ centered at $a=\frac{\pi}{2}$

3. Does Maclaurin's series of the functions $f(x)=\frac{1}{x}, g(x)=\operatorname{cosec} x$ and $h(x)=\sqrt{x}$ exist? If not why? Give appropriate justification.

4. Write MAPLE Command to find Taylor's Expression of the following functions:
   (i) $f(x) = e^x$ at $x = 1$ upto 10 terms.
   (ii) $f(x) = \sin x$ at $x = \pi$ upto 10 terms.
   (iii) $f(x) = \cos x$ at $x = \pi$ upto 10 terms.
   (iv) $f(x) = \ln(1 + x)$ at $x = 0$ upto 10 terms
   (v) $f(x) = \frac{1}{x}$ at $x = 1$ upto 5 terms.
   (vi) $f(x) = \frac{1}{x}$ at $x = 2$ upto 5 terms

4.3 Application of Derivatives

Derivatives have various important applications in Mathematics such as to find the Rate of Change of a Quantity, to find the Approximation Value, to find the equation of Tangent and Normal to a Curve, angle between two curves and to find the Minimum and Maximum Values of algebraic expressions. Derivatives are vastly used in the fields of science, engineering, physics, etc.

4.3.1 Give geometrical interpretation of derivative.

Let $P(x, y)$ be any point on the curve $y = f(x)$.

Referring to Figure 4.1, we have,

$y = f(x) = \overline{MP}$

$\delta x = \overline{MN} = \overline{PK}$

$y + \delta y = f(x + \delta x) = \overline{NQ}$

$\therefore \delta y = f(x + \delta x) - f(x) = \overline{KQ}$ ...(i)

For average rate of change, we divide both sides of equation (i), by δx,

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{\overline{KQ}}{\overline{PK}} = \tan \emptyset$$

= gradient (slope) of secant PQ

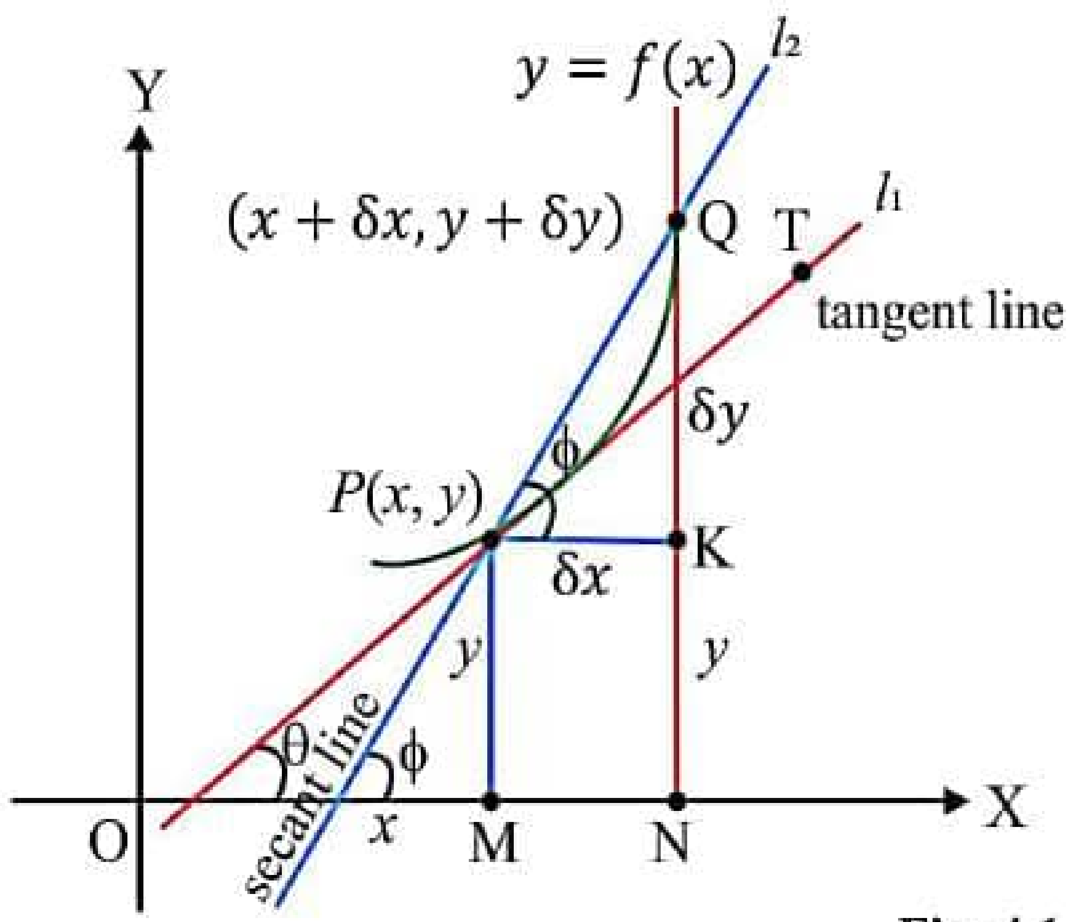


Fig. 4.1

Now, as δx approaches zero, the point Q will approach P along the curve, then secant will eventually becomes tangent

$$\text{So} \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

= gradient (slope) of the tangent PT = $\tan \theta$

where θ is the angle between the tangent at P and positive direction of x-axis.

Note: For derivative, $Q \to P \Rightarrow \tan \emptyset \to \tan \theta$.

i.e., $f'(x) = \frac{dy}{dx} = \tan\theta$, which is the slope of the tangent to the curve $y = f(x)$ at point.

We conclude that slope of the tangent to the curve is the derivative of the function of the curve at the point of tangency.

4.3.2 Find the equation of tangent and normal to the curve at a given point

Let $f(x, y) = 0$ is the equation of the curve. Then, to find the equation of the tangent at any given point (a, b) is found using following steps.

(i) Find slope of tangent at (a, b) i.e., $m = \left(\frac{dy}{dx}\right)_{(a,b)}$

(ii) By using point slope form, equation of tangent is $y - a = m(x - b)$.

Example 1. Find the equation of tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1)

Solution: Given curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

differentiate with respect to x regarding y as a function of x.

$$\therefore \quad \left(\frac{2}{3}x^{-\frac{1}{3}}\right) + \left(\frac{2}{3}y^{-\frac{1}{3}}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(x^{-\frac{1}{3}}\right) + \left(y^{-\frac{1}{3}}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Hence, the slope of the tangent at the point (1, 1) is $\left(\frac{dy}{dx}\right)_{(1,1)} = -1$

Now, by using point slope form of line

$y - 1 = -1(x - 1)$ or $y + x - 2 = 0$

To find the equation of normal, the slope of the normal at the point (1,1) is equal to negative reciprocal of the slope of tangent. Therefore, the slope of the normal is 1.

Hence, the equation of the normal is $y - 1 = 1(x - 1)$ or $y - x = 0$

Example 2. Find the equation of the tangent to the curve $y = \frac{(x-7)}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution: The equation of curve is $y = \frac{(x-7)}{(x-2)(x-3)}$ ...(i)

As curve cuts the x-axis, so $y = 0$.

Using $y = 0$ in (i) we get

$$0 = \frac{x-7}{(x-2)(x-3)} \qquad \Rightarrow x = 7$$

Thus, the point where the curve cuts x-axis is $(7, 0)$.

Now, differentiating (i) with respect to x, we get

$$y = \frac{(x-7)}{(x-2)(x-3)} = \frac{(x-7)}{(x^2-5x+6)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2-5x+6)-(x-7)(2x-5)}{(x^2-5x+6)^2} = \frac{x^2-5x+6-2x^2+19x-35}{(x^2-5x+6)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^2+14x-29}{(x^2-5x+6)^2}$$

$$\left[\frac{dy}{dx}\right]_{(7,0)} = \frac{-49+98-29}{(49-35+6)^2} = \frac{20}{400} = \frac{1}{20}$$

$\therefore$ slope of the tangent to the curve (i) at point $(7, 0)$ is $\frac{1}{20}$.

Equation of tangent is $y - y_1 = \frac{1}{20}(x - x_1)$

At (7,0) $\quad y - 0 = \frac{1}{20}(x-7) \Rightarrow 20y = x - 7$

$$\Rightarrow x - 20y - 7 = 0$$

4.3.3 Find the angle of intersection of the two curves.

Angle between two curves:

Let $y_1 = f(x)$ and $y_2 = g(x)$ be two curves which intersect each other at point $P(x_1, y_1)$ as shown in the figure 4.2. If we draw tangent line passing through intersecting point of the curves, then the angle between these tangent lines is called the angle between two curves.

To find the angle take m_1, m_2 be the slopes of tangent lines. By the definition of slope

$m_1 = \tan\alpha$

and $\quad m_2 = \tan\beta$

where α and β are the inclinations of the lines and can be calculated by using derivative.

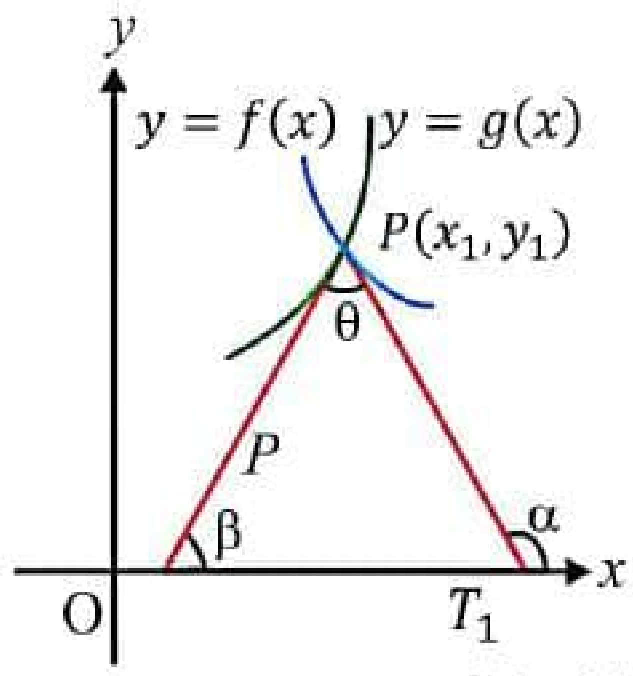


Fig. 4.2

The acute angle between the curves is given by

$$\theta = tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Steps to be followed to find the angle between two curves:

(i) Find point of intersection by solving the equations of both curves.

(ii) Find $\frac{dy}{dx}$ of both curves

(iii) Put value of point of intersection in $\frac{dy}{dx}$ and get m_1 and m_2.

(iv) Put value of m_1 and m_2 in $\theta = tan^{-1}\left|\frac{m_1-m_2}{1+m_1\,m_2}\right|$ and find θ.

Example: Find the angle between the curves $xy = 2$ and $y^2 = 4x$.

Solution: Given equations of curves are $xy = 2$ ..(i)

$y^2 = 4x$ ..(ii)

From equation (i) and (ii), we get

$$\frac{4}{x^2} = 4x$$

$$\Rightarrow \quad x^3 = 1 \qquad \Rightarrow x = 1$$

To get the value of y, put $x = 1$ in equation (i), we get $y = 2$, so the point of intersection of curves is $(1, 2)$.

Let m_1 be the slope of curve (i) at the point $(1,2)$.

By differentiating on both sides of equation (i) with respect to x we get,

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(1,2)} = \left(\frac{-2}{1}\right)_{(1,2)} = -2$$

Similarly, m_2 be the slope of curve (ii), at the point $(1,2)$ is given by

$$\frac{dy}{dx} = \frac{2}{y}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(1,2)} = \left(\frac{2}{y}\right)_{(1,2)} = 1$$

Angle between the given curves,

$$tan\,\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2 - 1}{1 + (-2)(1)}\right| = \left|\frac{-3}{-1}\right| = 3$$

Hence $\theta = \tan^{-1}(3) = 71.56°$

4.3.4 Find the point on a curve where the tangent is parallel to the given line

Example 1. Find the point on the curve $xy = 12$, the tangent at the point is parallel to the given line $3x + y = 3$.

Solution: The slope of the line $3x + y = 3$, is $m_1 = -3$

From the equation of curve $xy = 12$ so, $y = \frac{12}{x}$

By differentiating with respect to x to get $\frac{dy}{dx} = \frac{-12}{x^2} = m_2$

As the given line is parallel to tangent to the curve so $\boldsymbol{m_1} = \boldsymbol{m_2}$ or $-3 = \dfrac{-12}{x^2}$

$$\Rightarrow x^2 = 4 \qquad \text{or} \quad x = \pm 2$$

By substituting the value of x in the equation of curve $xy = 12$ we have

$$y = \frac{12}{2} = 6 \qquad \text{at } x = 2$$

$$y = \frac{12}{-2} = -6 \qquad \text{at } x = -2$$

The points at which the tangent line is parallel to the given line are $(2, 6)$ and $(-2, -6)$.

Exercise 4.3

1. Determine the slope of tangent to the curve $y = x^3$ at the point $\left(\frac{3}{2}, \frac{27}{8}\right)$.
2. Find the slope of tangents to the curve $x^2 + y^2 = 25$ at the point on it whose x–coordinate is 2.
3. Find the equation of the tangent and the equation of the normal to the curve $y = x + \frac{1}{x}$ at the point where $x = 2$.
4. Given two curves $y = x^2$ and $y = (x-3)^2$. Find the angle between them
5. Prove that the tangent lines to the curve $y^2 = 4ax$ at points where $x = a$ are at right angles to each other.
6. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to y-axis.

4.4 Maxima and Minima

4.4.1 Define increasing and decreasing functions

Maximum and minimum values of function are called maxima and minima of the function.

A function $f(x)$ is said to be increasing at a point $x = a$,

if $f(a-h) < f(a) < (a+h)$, where h is a positive change in x. (Fig. 4.3)

A function $f(x)$ is said to be decreasing at a point $x = a$, if

$$f(a-h) > f(a) > f(a+h) \text{ (Fig. 4.4)}$$

A function is said to be increasing or decreasing over an interval if it is increasing or decreasing at every point of that interval.

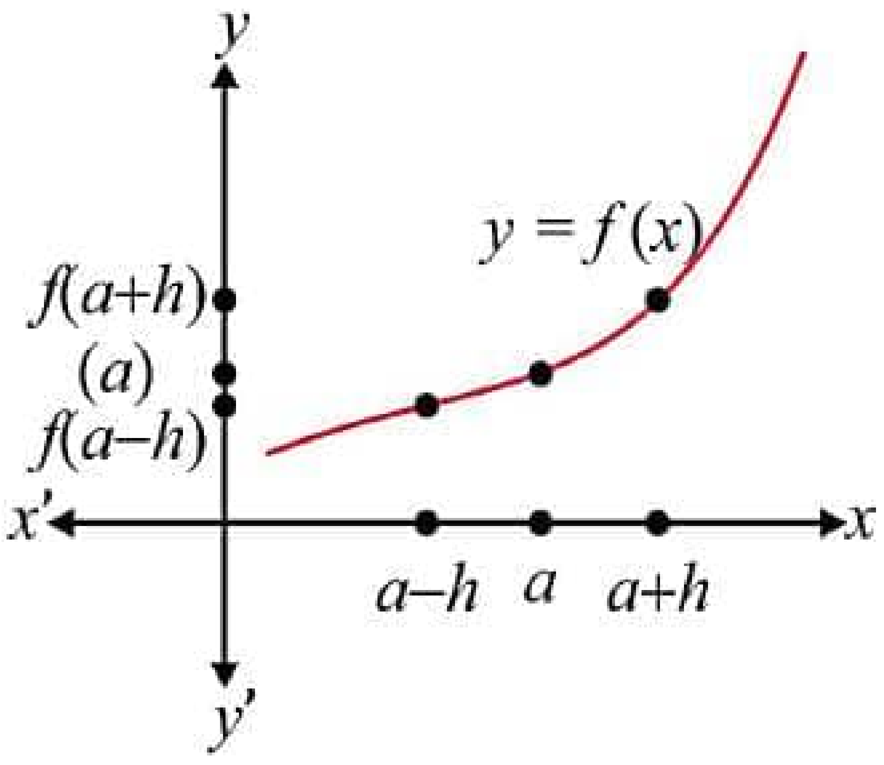


Fig. 4.3

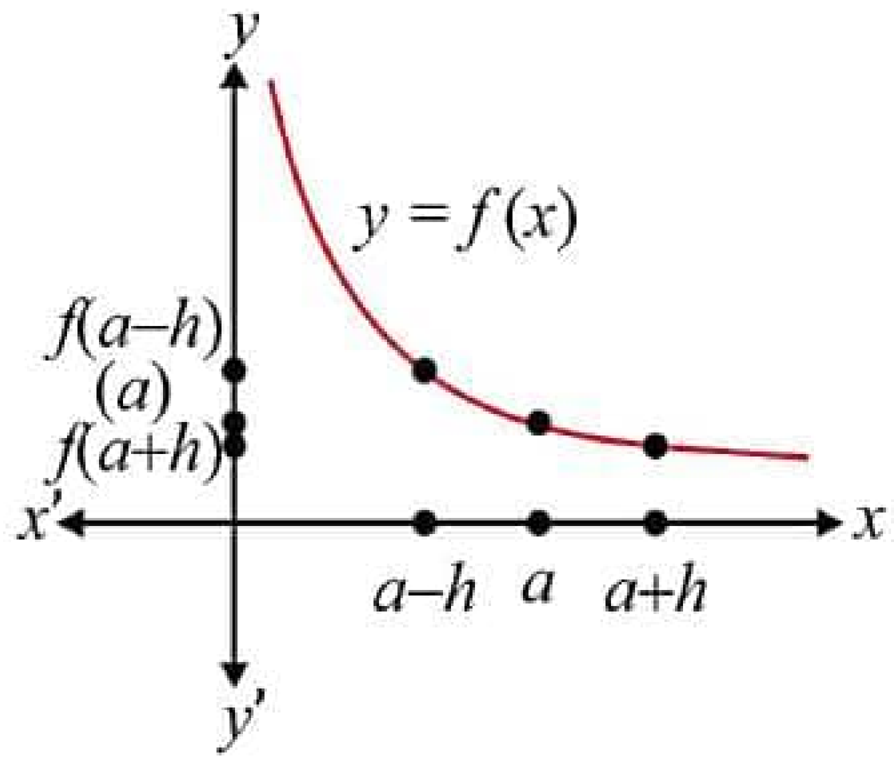


Fig. 4.4

The Fig. 4.3, represents the increasing function and the Fig. 4.4 represents the decreasing function.

4.4.2 Prove that if $f(x)$ is differentiable function on the open interval (a, b) then

- $f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall x \in (a, b)$
- $f(x)$ is decreasing on (a, b) if $f'(x) < 0\ \forall x \in (a, b)$

- **$f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall x \in (a, b)$**

Let $f(x)$ is increasing function at x where $x \in (a, b)$, then by the definition of increasing function.

$f(x+h) > f(x)$, where h is positive change in x.

Now, by definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\because \quad f(x+h) - f(x) > 0$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} > 0$$

Hence, the function $f(x)$ is increasing on (a, b) if $f'(x) > 0\ \forall\ x \in (a, b)$.

- **$f(x)$ is decreasing on (a, b) if $f'(x) < 0\ \forall x \in (a, b)$**

Let $f(x)$ is decreasing function at x where $x \in (a, b)$, then by the definition of decreasing function.

$f(x+h) < f(x)$, where h is positive change in x.

Now, by definition of derivative,

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\because \quad f(x+h) - f(x) < 0$

$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} < 0$

Hence, the function $f(x)$ is decreasing on (a, b) if $f'(x) < 0 \; \forall \; x \in (a, b)$.

Example 1. Check whether the function $f(x) = x^2 + 5$ is increasing at $x = 3$ or not.

Solution:

$$f(x) = x^2 + 5$$

Differentiating w.r.t x,

We get

$$f'(x) = 2x$$

Put $x = 3$,

$$f'(3) = 2(3)$$

$$f'(3) = 6 > 0$$

Hence, the function $f(x) = x^2 + 5$ is increasing at $x = 3$.

Example 2. Check whether $y = \sin x$ is decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution:

$$f(x) = \sin x$$

Differentiating w.r.t to x

We get

$$f'(x) = \cos x$$

$\because \quad \cos x < 0 \; \forall \; x \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \quad f(x) = \sin x$ is decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

4.4.3 Examine a given function for extreme values.

Let $y = f(x)$ be a function, and the graph of this function be shown in Fig. 4.5.

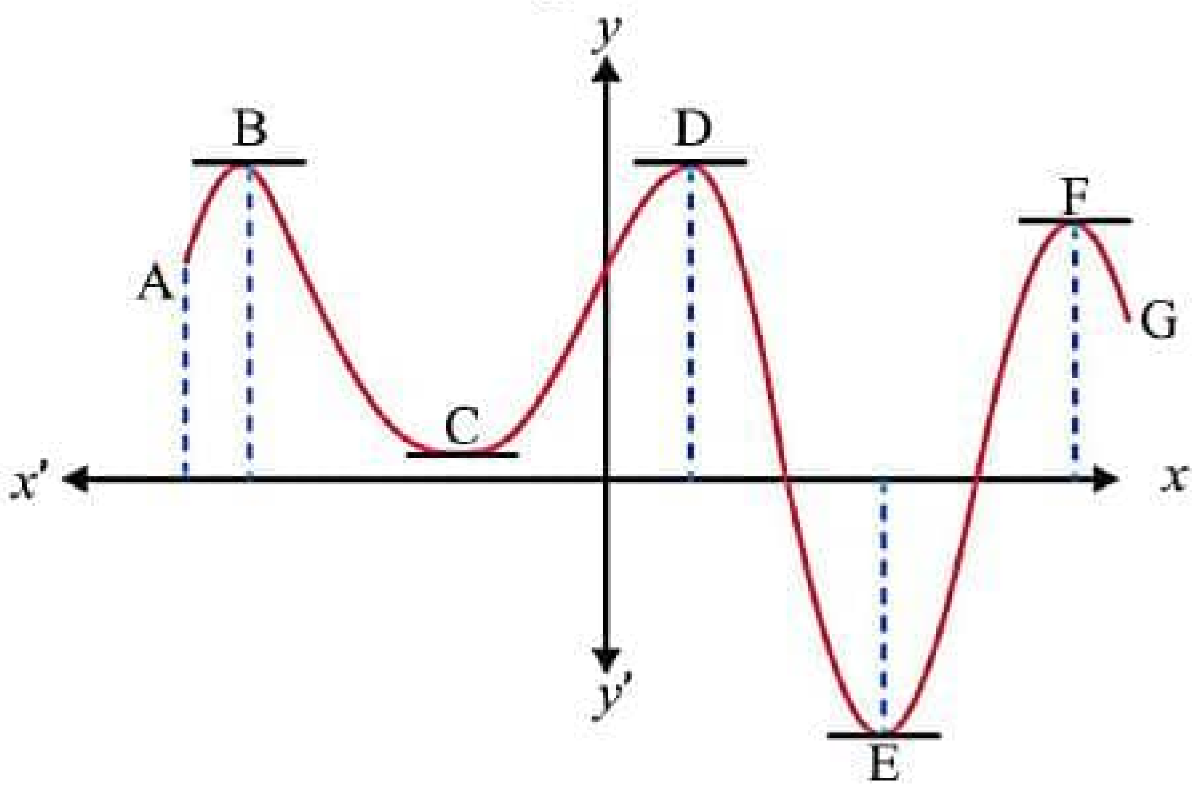


Fig. 4.5

From diagram F is increasing from A to B, C to D, E to F and d21ecreasing from B to C, D to E and F to G.

Points B, C, D, E and F are such points where the function is neither increasing nor decreasing. The tangents to the curve at these points are parallel to x-axis. The derivative of the function at these points vanishes, these points are called turning points or points of extreme values or extrema.

B, D and F are such turning points where the function changes from increasing to decreasing. These are called points of maximum values or maxima.

C and E are such turning points where the function changes from decreasing to increasing. These are called points of minimum values or minimas.

Maxima and Minima through first derivative

Let $y = f(x)$ be a function.

(i) Differentiate w.r.t 'x' and obtain $f'(x)$.

(ii) Put $f'(x) = 0$, solve it and obtain critical points.

(iii) Let $x = a$ be a critical point.

If $\left.\begin{array}{l} f'(a-h) < 0 \\ f'(a) = 0 \\ f'(a+h) > 0 \end{array}\right\} \Rightarrow x = a$ is point of minima.

If $\left.\begin{array}{l} f'(a-h) > 0 \\ f'(a) = 0 \\ f'(a+h) < 0 \end{array}\right\} \Rightarrow x = a$ is point of maxima.

If $\left.\begin{array}{l} f'(a-h) > 0 \\ f'(a) = 0 \\ f'(a+h) > 0 \end{array}\right\}$ or If $\left.\begin{array}{l} f'(a-h) < 0 \\ f'(a) = 0 \\ f'(a+h) < 0 \end{array}\right\} \Rightarrow x = a$ is point of inflection.

Note: Point of inflection is that point of curve which is nether point of minimum nor maximum.

4.4.4 State the second derivative rule to find the extreme values of a function at a point

Let $y = f(x)$ be a function.

(i) Differentiate w.r.t 'x' and obtain $f'(x)$.

(ii) Put $f'(x) = 0$, solve it and obtain critical function.

(iii) Differentiate again w.r.t 'x' of obtain $f''(x)$.

(iv) Let $x = a$ be a critical point.

If the $f''(a) < 0 \Rightarrow x = a$ is a point of maxima.

If the $f''(a) > 0 \Rightarrow x = a$ is point of minima.

If the $f''(a) = 0 \Rightarrow$ test fails.

4.4.5 Use second derivative rule to examine a given function for extreme values.

Example 1. Find extreme values of $f(x) = x^4 - 8x^2$ using the second derivative rule.

Solution: Here $f(x) = x^4 - 8x^2$

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$$

Put $f'(x) = 0$

$4x(x^2 - 4) = 0$

$\Rightarrow$ $x = 0$ or $x = \pm 2$

Again Differentiate

$f''(x) = 12x^2 - 16$

Putting the values of $x = -2, 0$ and 2 into $f''(x)$.

$f''(-2) = 12(-2)^2 - 16 = 32 > 0$ that is function has a minimum at $x = -2$

$f''(0) = 12(0)^2 - 16 = -16 < 0$ function has a maximum at $x = 0$

$f''(2) = 12(2)^2 - 16 = 32 > 0$ function has a minimum at $x = 2$

Minimum value at $x = -2$

$f(-2) = (-2)^4 - 8(-2)^2 = -16$

Minimum value at $x = 2$

$f(2) = (2)^4 - 8(2)^2 = -16$

Maximum value at $x = 0$

$f(0) = 0$

Example 2. Find points of extrema of $f(x) = \sin x + \cos x$ on $[0,2\pi]$ using the Second Derivative Rule.

Solution: $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

As $f'(x) = 0$

We have $\cos x - \sin x = 0$

$\sin x = \cos x$

Dividing both side by cos x

$\tan x = 1$, $x = tan^{-1}(1) = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

so in the interval $[0,2\pi]$ we have $f'(x) = 0$ at $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

Again Differentiate

$$f''(x) = -\sin x - \cos x$$

Put the values of $x = 0, \frac{\pi}{4}, \frac{5\pi}{4}, 2\pi$

$f''(0) = -\sin(0) - \cos(0) = -1 < 0$ that is function has a maximum at $x = 0$

$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$ function has a maximum at $x = \frac{\pi}{4}$

$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$ function has a minimum at $\frac{5\pi}{4}$

$f''(2\pi) = -\sin(2\pi) - \cos(2\pi) = -1 < 0$ function has a maximum at 2π.

4.4.6 Solve real life problems related to extreme value.

Example 1. A farmer wishes to enclose a rectangular field using 1000 yards of fencing in such a way that the area of the field is maximized.

Solution: Let x and y be the length and breadth of the field and A be the area of the field. then $A = xy$.

For fencing we have an equation for perimeter

$2x + 2y = 1000, \Rightarrow \quad y = 500 - x$

Area of rectangular field

$A = x(500 - x) = 500x - x^2$

Now $\frac{dA}{dx} = 500 - 2x,$

so $\frac{dA}{dx} = 0,$

$500 - 2x = 0, \quad x = 250$

when $x = 250.$

$\frac{d^2A}{dx^2} = -2$

Hence $\left(\frac{d^2A}{dx^2}\right)_{x=250} = -2 < 0$

The maximum area occurs at $x = 250$.

i.e., $A = 250(500 - 250) = 62500$ square and dimension of rectangular field is $x = 250$ and $y = 250$.

Example 2. A company finds that the cost of goods $C(x)$ is given by

$$C(x) = -x^3 + 9x^2 - 15x + 9$$

where x represents thousand of units. If the company can only make a minimum of 6000 units, what is the minimum cost company required. Here, cost is in dollar.

Solution: since x is in thousand of unit we must find the minimum cost in the interval $[0,6]$

$C'(x) = -3x^2 + 18x - 15$

$= -3(x^2 - 6x + 5)$

$= -3(x-5)(x-1) = 0$

$\Rightarrow \quad x = 5, 1$

$\Rightarrow \quad C''(x) = -6x + 18$

$C''(1) = -6x + 18 = 12 > 0$

$C''(5) = -6(5) + 18 = -12 < 0$

i.e., $\quad C(1) = 12.$

The minimum cost for the company exist at $x = 1$. i.e., $C(1) = 12$.

4.4.7 Use MAPLE command Maximize (Minimize) to compute maximum (minimum) value of a function

The format of Maximize (Minimize) command in MAPLE is as under:

```
> maximize(f(x), x=a..b)
> minimize(f(x), x=a..b)
```

where,

f(x) is the function whose maximize (minimize) value is required

x = a..b is the interval for maximize (minimize) value

In order to compute the Maximize (Minimize) value of a function in the interval, following examples are given:

$> minimize\,(cos(x), x = -\pi..\pi)$

-1

$> maximize\left(cos(x), x = \frac{-\pi}{2}..\frac{\pi}{2}\right)$

1

$> minimize(sin(x), x = -\pi..\pi)$

-1

$> maximize\left(sin(x), x = \frac{-\pi}{2}..\frac{\pi}{2}\right)$

1

$> minimize(exp(x), x = 0..10);$

1

$> maximize(exp(x), x = 0..10);$

$exp(10)$

$> minimize(exp(x), x = 0..5);$

$exp(1)$

$> maximize(exp(x), x = 0..5);$

$exp(5)$

$> maximize\,(x^2 + y^2 - 2x + 2y + 2), x = 1..2, y = -2..2$, location);

$10, \{[\{x = 2, y = 2\}, 10]\}$

$> minimize\,(x^2 + y^2 - 2x + 2y + 2), x = 1..2, y = -2..2$, location);

$10, \{[\{x = 2, y = 2\}, 0]\}$

Exercise 4.4

1. Show that function $f(x) = -x^2 + 10 + 9$ is increasing at $x = 4$.
2. Show that $f(x) = \tan^2 x$ is decreasing at $x = \frac{3\pi}{4}$.
3. Find the maximum and minimum values, if any, of the function $f: \mathbb{R} \rightarrow \mathbb{R}$ in the following cases:
   (i) $f(x) = x^2 - 2x + 3$ (ii) $f(x) = x^2 - 9x^2 + 15x + 3$
   (iii) $f(x) = -x^4 + 2x^2$ (iv) $f(x) = e^x \sin x$
   (v) $f(x) = 2e^x + e^{-x}$ (vi) $f(x) = 2x - x^2$
4. A rectangular reservoir with a square bottom and open top is to be lined inside with lead. Find the dimensions of the reservoir to hold $\frac{1}{2}a^3$ cubic metres, such that the lead required is minimum.
5. Find a right-angled triangle of maximum area with a hypotenuse of length h.
6. A particle moves so that its distance s at time is given by $s = ut + \frac{1}{2}at^2$, where u a are fixed real numbers. Find its speed and magnitude of its accelerations at time t.
7. If the P and time t are connected by the formula $P = t^2 + 11 + 9$. Find force at $t = 5$ second.

Review Exercise 4

1. Choose the correct answer

(i) If $y = \frac{1}{x}$, then $y'''(1) =$ --------------

(a) 6 (b) -6 (c) 2 (d) None of these

(ii) What will be the nth derivative of $2e^x$

(a) $2ne^x$ (b) $2e^{nx}$ (c) $2e^x$ (d) $\frac{e^x}{2}$

(iii) If $xy = k^2$, then $y'' =$ __

(a) $\frac{2k^2}{x}$ (b) $\frac{-2k^2}{x^3}$ (c) $\frac{3k^2}{x^3}$ (d) $\frac{2k^2}{x^3}$

(iv) 3rd order derivative of 2^x is ---------------

(a) $(ln3)^2\, e^{xln2}$ (b) $(ln2)^3\, e^{xln3}$ (c) $(ln2)^3\, e^{xln2}$ (d) None of these

(v) 2nd order derivative of $f(x) = A\sin x + B\cos x$ is -------------

(a) $f(x)$ (b) $-f(x)$ (c) $\pm f(x)$ (d) None of these

(vi) If S is the distance covered by a car, then $\frac{d^2S}{dt^2}$ will be its -----------------

(a) Velocity (b) deceleration (c) acceleration (d) Average velocity

(vii) $y = \tan^{-1}x$, then $y''(1) =$ -----------------

(a) $\frac{-1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{-1}{4}$ (d) $\frac{1}{2}$

(viii) If $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $x \in R$, then y is the McLaurin series of

(a) $\sin x$ (b) $\cos x$ (c) e^x (d) None of these

(ix) A function $f(x)$ is said to be a decreasing function when $x_1 < x_2$ and

(a) $f(x_1) < f(x_2)$ (b) $f(x_1) > f(x_2)$

(c) $f(x_1) \geq f(x_2)$ (d) $f(x_1) \leq f(x_2)$

(x) If a function $f(x)$ is such that $f'(c) = 0$ then the point $(c, f(c))$ is called

(a) Maximum point (b) Minimum point

(c) Stationary point (d) Critical point

2. Find 2nd order derivative of $f(x) = \ln(1 + x^2)$

3. Find the second derivative of following parametric functions

$x = 3u^2 + 1$ and $y = 3u^2 + 5u$, Find $\frac{d^2y}{dx^2}$

4. Evaluate the third derivatives of the given functions.

(i) $v(x) = x^3 - x^2 + x - 1$

(ii) $f(x) = 7\sin\left(\frac{x}{3}\right) + \cos(1 - 2x)$

(iii) $y = e^{-5x} + 8\ln(2x^4)$

5. Determine the second derivative of the given functions.

(i) $g(x) = \sin(2x^3 - 9x)$ (ii) $z(x) = \ln(7 - x^3)$

(iii) $q(x) = \frac{2}{(6+2x-x^2)^4}$ (iv) $h(t) = cos^2(7t)$

(v) $2x^3 + y^2 = 1 - 4y$ (vi) $6y - xy^2 = 1$

6. Given functions

(i) $y = \cos x$; Find y'; y''; y'''; $y^{(4)}$;

7. Find $\frac{d^2y}{dx^2}$; if $x = 4\sin t$, $y = 5\cos t$.

8. Find the rth derivative of $f(x) = x^n$ where $r \leq n$.

9. Find the Taylor series of the function $x^4 + x - 2$ centered at $a = 1$.

10. Obtain the Taylor series for the function $(x-1)e^x$ near $x=1$.
11. Find the McLaurin series for $\ln(1+x)$ and hence find for $\ln\left(\frac{1+x}{1-x}\right)$.
12. Find the equation of the tangent line to the curve $y=x^3-3x^2+x$ at the point $(2,-2)$.
13. At what point on the graph of $y=x^2$ where the tangent line is parallel to the line $3x-y=2$.
14. Determine the interval on which the function $f(x)=x^2-3x+1$ is increasing and decreasing.

Unit 5 Differentiation of Vector Functions

5.1 Scalar and Vector Functions

5.1.1 Define scalar and vector function

Scalar function:

A scalar function is a function whose domain and codomain are the subsets of real number.

For example, area of circle is the scalar function of its radius which is defined as $A = \pi r^2$ and temperature is the scalar function of time.

Vector function:

A vector function is a function where each real number in the domain is mapped to either a two or three-dimensional vector. It is donated as $\vec{r}(t)$.

Mathematically, it is written as

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

where $f(t), g(t)$ and $h(t)$ are the components of the vector and they are scalar functions of variable t.

Examples include velocity and acceleration are the vector functions of time.

Let $\vec{F}(t)$ be a vector function. If the initial point of the vector $F(t)$ is at the origin, then the graph of vector $\vec{F}(t)$ is the curve traced out by the terminal point of the position vector $\vec{F}(t)$ as t varies over the domain set D. This is shown in the figure 5.1.

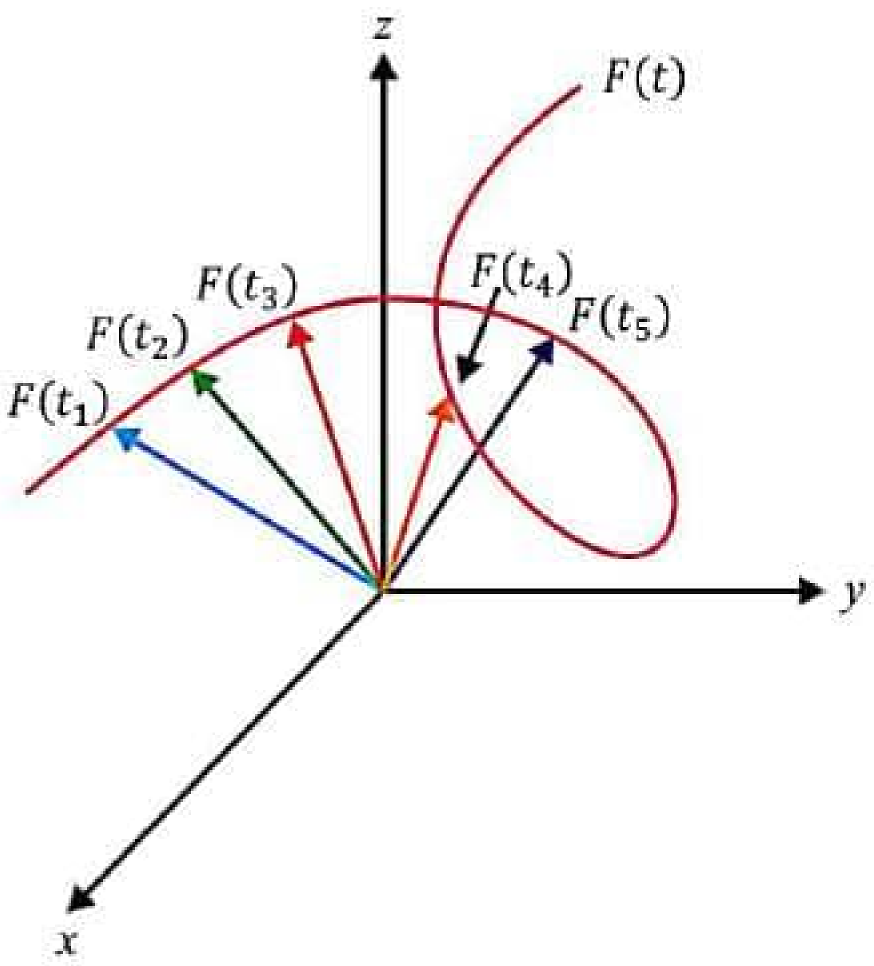


Fig. 5.1

5.1.2 Explain domain and range of a vector function

The domain of the vector function is the set of real numbers and the range of the vector function is the set of the vectors. According to the definition of vector function, it is written as

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Hence it is function of variable t which is scalar quantity. Therefore, the domain is the set of real numbers. However, the output of the function is a vector. So, its range is the set of vectors.

The intersection of the domains of each components of vector function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ is the domain of $\vec{r}(t)$.

i.e., Dom $\vec{r}(t) =$ Dom $f(t) \cap$ Dom $g(t) \cap$ Dom $h(t)$

Example: Find the domain for the following vector function $\vec{r}(t) = t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t+3)\hat{k}$.

Solution: The vector function is $\vec{r}(t) = t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t+3)\hat{k}$

here $f(t) = t^2, g(t) = \frac{1}{t}$ and $h(t) = t + 3$

Dom $f = \mathbb{R}$, Dom $g = \mathbb{R} - \{0\}$, Dom $h = \mathbb{R}$

$\Rightarrow$ Dom $\vec{r} = \mathbb{R} - \{0\}$

5.2 Limit and Continuity

5.2.1 Define limit of a vector function and employ the usual technique for algebra of limits of scalar function to demonstrate the following properties of limits of a vector function.

- **The limit of the sum (difference) of two vector functions is the sum (difference) of their limits.**
- **The limit of the dot product of two vector functions is the dot product of their limits.**
- **The limit of the cross product of two vector functions is the cross product of their limits.**
- **The limit of the product of a scalar function and a vector function is the product of their limits.**

Limit of a vector function:

Limit of vector function $\vec{r}(t)$ at $t = t_0$ is the vector $\vec{L}$, such that the values of vector function get close to $\vec{L}$ as long as t becomes close enough to t_0.

i.e., $\lim\limits_{t\to t_o} \vec{r}(t) = \vec{L}$

The limit of $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ exists at $t = t_0$ if limit of each component of vector function $f(t), g(t)$ and $h(t)$ exists at t_0.

To obtain the limit of $\vec{r}(t)$ at $t = t_0$

Let $\lim\limits_{t\to t_0} \vec{r}(t) = a, \lim\limits_{t\to t_0} g(t) = b$ and $\lim\limits_{t\to t_0} h(t) = c$

then $\lim\limits_{t\to t_0} \vec{r}(t) = \lim\limits_{t\to t_0} \left(f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right)$

$= a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Example 1. Find the limit of vector function $\vec{r}(t) = \frac{e^t-1}{t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}$ when $t \to 0$

Solution: Here, $\vec{r}(t) = \frac{e^t-1}{t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}$

Now, $\lim\limits_{t\to 0} \vec{r}(t) = \lim\limits_{t\to 0}\left(\frac{e^t-1}{e^t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}\right)$

$$= \left(\lim_{t\to 0} \frac{e^t - t}{e^t}\right) \hat{\imath} + \left(\lim_{t\to 0} \frac{\sqrt{1+t}-1}{t}\right) \hat{\jmath} + \left(\lim_{t\to 0} \frac{3}{1+t}\right) \hat{k}$$

$$= \left(\frac{1-0}{1}\right) \hat{\imath} + \left(\lim_{t\to 0} \frac{\sqrt{1+t}-1}{t} \cdot \frac{\sqrt{1+t}+1}{\sqrt{1+t}+1}\right) \hat{\jmath} + \left(\frac{3}{1+0}\right) \hat{k}$$

$$= \hat{\imath} + \left(\lim_{t\to 0} \frac{t}{t\left(\sqrt{1+t}+1\right)}\right) \hat{\jmath} + 3\hat{k}$$

$$= \hat{\imath} + \left(\lim_{t\to 0} \frac{1}{\sqrt{1+t}+1}\right) \hat{\jmath} + 3\hat{k} \;=\; \hat{\imath} + \left(\frac{1}{\sqrt{1+0}+1}\right) \hat{\jmath} + 3\hat{k}$$

$$= \hat{\imath} + \frac{1}{2} \hat{\jmath} + 3\hat{k}$$

The limit of the Sum (difference) of two vector functions is the sum of their limits

Limit of the sum or difference of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the sum or difference of the limits of each vector function.

i.e., $$\lim_{t\to t_o} [\vec{r}(t) \pm \vec{s}(t)] = \lim_{t\to t_o} \vec{r}(t) \pm \lim_{t\to t_o} \vec{s}(t)$$

The limit of the dot product of two vector functions is the dot product of their limit functions:

Limit of the dot product of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the dot product of their limits.

i.e., $$\lim_{t\to t_o} [\vec{r}(t) \cdot \vec{s}(t)] = \left[\lim_{t\to t_o} \vec{r}(t)\right] \cdot \left[\lim_{t\to t_o} \vec{s}(t)\right]$$

The limit of the cross product of two vector functions is the cross product of their limits:

Limit of the cross product of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the cross product of the limits of each vector function.

i.e., $$\lim_{t\to t_o} [\vec{r}(t) \times \vec{s}(t)] = \left[\lim_{t\to t_o} \vec{r}(t)\right] \times \left[\lim_{t\to t_o} \vec{s}(t)\right]$$

The limit of the product of a scalar function and a vector function is the product of their limits:

Limit of the product of a scalar function $h(t)$ and a vector function $\vec{s}(t)$ is the product of their limits.

i.e., $$\lim_{t\to t_o} [h(t)\, \vec{s}(t)] = \left(\lim_{t\to t_o} h(t)\right) \left[\lim_{t\to t_o} \vec{s}(t)\right]$$

Example 2. If $\vec{u} = t^3\hat{\imath} - 3\hat{\jmath}$; $\vec{v} = 3t^2\hat{\imath} - \hat{k}$ are vector functions and $h(t) = t + 3$ is scalar function then find the following:

(i) $\lim_{t\to 3} [\vec{u}(t) - \vec{v}(t)]$ (ii) $\lim_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$

(iii) $\lim_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim_{t\to 0} [h(t)\, \vec{u}(t)]$

Solution:

(i) $\lim_{t\to3}[\vec{u}(t)-\vec{v}(t)] = \left[\lim_{t\to3}\vec{u}(t)\right]-\left[\lim_{t\to3}\vec{v}(t)\right]$

$$= \left[\lim_{t\to3}(t^3\hat{\imath}-3\hat{\jmath})\right]-\left[\lim_{t\to3}(3t^2\hat{\imath}-\hat{k})\right]$$
$$= [(3)^3\hat{\imath}-3\hat{\jmath}]-[3(3)^2\hat{\imath}-\hat{k}]$$
$$= [27\hat{\imath}-3\hat{\jmath}]-[27\hat{\imath}-\hat{k}] = 27\hat{\imath}-27\hat{\imath}-3\hat{\jmath}+\hat{k}$$
$$= -3\vec{j}+\vec{k}$$

(ii) $\lim_{t\to1}[\vec{u}(t)\cdot\vec{v}(t)] = \left[\lim_{t\to1}\vec{u}(t)\right]\cdot\left[\lim_{t\to1}\vec{v}(t)\right]$

$$= \left[\lim_{t\to1}(t^3\hat{\imath}-3\hat{\jmath})\right]\cdot\left[\lim_{t\to1}(3t^2\hat{\imath}-\hat{k})\right]$$
$$= [(1)^3\hat{\imath}-3\hat{\jmath}]\cdot[3(1)^2\hat{\imath}-\hat{k}] = [\hat{\imath}-3\hat{\jmath}]\cdot[3\hat{\imath}-\hat{k}] \quad \because[\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1]$$
$$= (1)\cdot(3)(\hat{\imath}\cdot\hat{\imath})+(-3)\cdot(0)(\hat{\jmath}\cdot\hat{\jmath})+(0)\cdot(-1)(\hat{k}\cdot\hat{k}) = 3$$

(iii) $\lim_{t\to1}[\vec{u}(t)\times\vec{v}(t)] = \left[\lim_{t\to1}\vec{u}(t)\right]\times\left[\lim_{t\to1}\vec{v}(t)\right]$

$$= \left[\lim_{t\to1}(t^3\hat{\imath}-3\hat{\jmath})\right]\times\left[\lim_{t\to1}(3t^2\hat{\imath}-\hat{k})\right]$$
$$= [(1)^3\vec{\imath}-3\vec{\jmath}]\times[3(1)^2\hat{\imath}-\hat{k}]$$
$$= [\hat{\imath}-3\hat{\jmath}]\times[3\hat{\imath}-\hat{k}]$$
$$= \begin{vmatrix} i & j & k \\ 1 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix} = [(-3)(-1)-0]\vec{\imath}-[(1)(-1)-0]\vec{\jmath}+[(1)(0)-(-3)(3)]\vec{k}$$
$$= 3\vec{\imath}+\vec{\jmath}+9\vec{k}$$

(iv) $\lim_{t\to0}[h(t)\,\vec{u}(t)] = \left[\lim_{t\to0}h(t)\right]\left[\lim_{t\to0}\vec{u}(t)\right]$

$$= \left[\lim_{t\to0}(t+3)\right]\left[\lim_{t\to0}(t^3\hat{\imath}-3\hat{\jmath})\right] = 3\left(\left[\lim_{t\to0}t^3\right]\hat{\imath}-\left[\lim_{t\to0}3\right]\hat{\jmath}\right)$$
$$= -9\hat{\jmath}$$

5.2.2 Define continuity of a vector function and demonstrate through examples

A vector function $\vec{r}(t) = f(t)\,\hat{\imath}+g(t)\,\hat{\jmath}+h(t)\hat{k}$ is continuous at $t = t_o$ if the following conditions are satisfied

(i) $t = t_o$ belongs to the domain of a vector function $\vec{r}(t)$

(ii) $\vec{r}(t_o) = \lim_{t\to t_o}\vec{r}(t) = \vec{L}$

It means value of the vector function $\vec{r}(t)$ at $t = t_o$ is equal to limit of the vector function when t approaches t_o.

If a vector function is continuous at a point then its all components will be continuous at that point.

Example 1. Show that the function $\vec{G}(t) = e^t\hat{\imath} + \cos t\ \hat{\jmath}$ is continuous at $t = 0$

Solution: The components of vector function are $f(t) = e^t$; $g(t) = \cos t$;

At $t = 0$ $\quad f(0) = e^0 = 1$; $\quad g(0) = \cos 0 = 1$

Now, $\quad \lim\limits_{t\to 0} \vec{G}(t) = \lim\limits_{t\to 0} (f(t)\,\hat{\imath} + g(t)\,\hat{\jmath})$

$$= \left(\lim_{t\to 0} f(t)\right)\hat{\imath} + \left(\lim_{t\to 0} g(t)\right)\hat{\jmath} = \left(\lim_{t\to 0} e^t\right)\hat{\imath} + \left(\lim_{t\to 0} \cos t\right)\hat{\jmath}$$

$$= e^0\,\hat{\imath} + \cos 0\ \hat{\jmath} = \hat{\imath} + \hat{\jmath}$$

Hence, $\quad \vec{G}(0) = \lim\limits_{t\to 0} \vec{G}(t)$

So, the vector function $\vec{G}(t)$ is continuous at $t = 0$. Hence shown

Example 2. Show that function $\vec{r}(t) = |t|\hat{\imath} + \frac{1}{t+1}\hat{\jmath}$ is continues at $t = 0$.

Solution: The components of vector function are

$f(t) = |t|\hat{\imath} + \frac{1}{t+1}\hat{\jmath}$

Now, $\quad \vec{r}(0) = |0|\hat{\imath} + \frac{1}{0+1}\hat{\jmath} = \hat{\jmath}$

We find limit of function

$$\lim_{t\to 0}(\vec{r}(t)) = \lim_{t\to 0}\left(|t|\,\hat{\imath} + \frac{1}{t+1}\hat{\jmath}\right) = \hat{\jmath}$$

$\because \quad \lim\limits_{t\to 0} \vec{r}(t) = \vec{r}(0)$

$\therefore \quad$ Function is continuous at $t = 0$.

Example 3. Test the continuity of $\vec{r}(t) = \frac{\hat{\imath}}{t^2} + 2t\hat{\jmath} + 3\hat{k}$ at $t = 1$.

Solution: $\quad \vec{r}(1) = \frac{\hat{\imath}}{(1)^2} + 2(1)\hat{\jmath} + 3\hat{k} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$$\lim_{t\to 1}\vec{r}(t) = \lim_{t\to 1}\left(\frac{\hat{\imath}}{t^2} + 2t\hat{\jmath} + 3\hat{k}\right) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} = \vec{r}(t)$$

$\vec{r}(t)$ is continuous at $t = 1$.

Exercise 5.1

1. Find the domain of the following vector function.

(i) $\vec{r}(t) = 2t\hat{\imath} - 3t\hat{\jmath} + \frac{1}{t}\hat{k}$ $\qquad$ (ii) $\vec{r}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \tan t\,\hat{k}$

(iii) $\vec{r}(t) = (1-t)\hat{\imath} + \sqrt{t}\hat{\jmath} + \frac{1}{t^2}\hat{k}$ $\qquad$ (iv) $\vec{g}(t) = \cos t\,\hat{\imath} - \cot t\,\hat{\jmath} + \operatorname{cosec} t\,\hat{k}$

2. Find the limit of vector function $\vec{r}(t) = (e^{3t} - 1)\,\hat{\imath} + \frac{\sqrt{3+t}-\sqrt{3}}{3t}\,\hat{\jmath} + \frac{1}{9t+1}\hat{k}$ at $t = 0$.

3. If $\vec{u} = t^2\hat{\imath} - 2\hat{\jmath}$; $\vec{v} = 2t\hat{\imath} - 5\hat{k}$ are vector functions and $h = 3t$ is scalar function then find the following:

(i) $\lim_{t\to 0} [\vec{u}(t) + \vec{v}(t)]$ (ii) $\lim_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$

(iii) $\lim_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim_{t\to 5} [h\,\vec{u}(t)]$

4. Show that function $\vec{R}(t) = sin^2 t\,\hat{\imath} + \tan t\,\hat{\jmath} + \frac{1}{t}\,\hat{k}$ is continuous at $t = \frac{\pi}{4}$.

5. Show that function $\vec{r}(t) = \frac{2}{t}\,\hat{\imath} + \frac{t^3}{2t^3-5}\,\hat{\jmath} + \frac{1}{e^t}\,\hat{k}$ is continuous at $t \to \infty$.

6. For what value of t, following vector functions are continuous

(i) $\vec{r}(t) = \ln(t+3)\,\hat{\imath} + \frac{1}{t-1}\,\hat{\jmath} + \frac{t+2}{t^2-4}\hat{k}$ (ii) $\vec{r}(t) = \frac{1}{3t+1}\hat{\imath} + \frac{1}{t}\,\hat{\jmath}$

5.3 Derivative of Vector Function

5.3.1 Define derivative of a vector function of a single variable and elaborate the result:

If $\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$, where $f_1(t), f_2(t), f_3(t)$ are differentiable functions of a scalar variable then

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{\imath} + \frac{df_2}{dt}\hat{\jmath} + \frac{df_3}{dt}\hat{k}$$

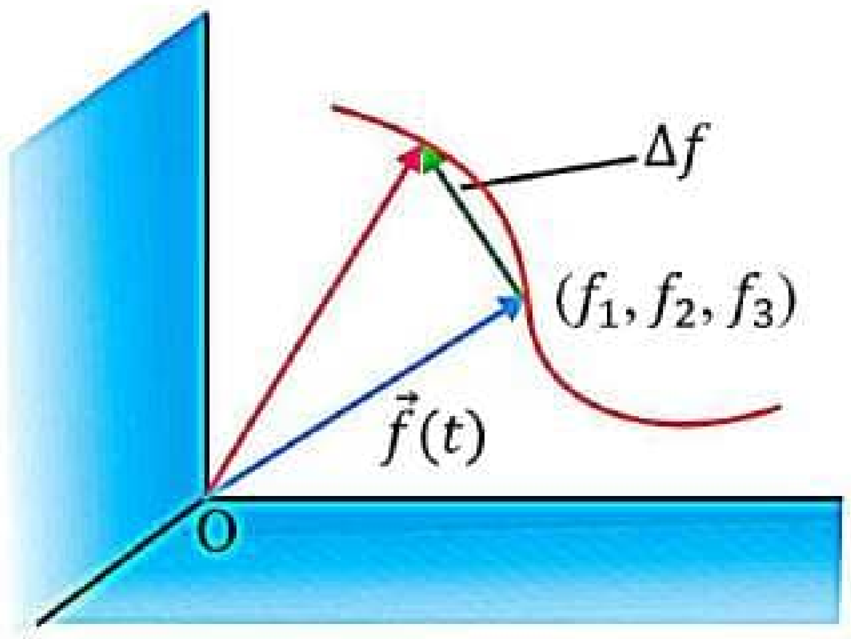


Fig. 5.2

Consider a vector function $\vec{f}(t)$ which is a curve, as the position vector function $\vec{f}(t)$ joining the origin O of a coordinate system at any point (f_1, f_2, f_3), then

$$\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Where, $f_1(t)$, $f_2(t)$, $f_3(t)$ are single variable scalar functions. As t changes, the vector function describes a curve having the following parametric equations.

$$f_1 = f_1(t),\ \ f_2 = f_2(t),\ \ f_3 = f_3(t)$$

Thus $\lim_{\Delta t\to 0} \frac{\Delta\vec{f}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$

is a vector in the direction of $\Delta\vec{f}$. If $\lim_{\Delta t\to 0} \frac{\Delta\vec{f}(t)}{\Delta t} = \frac{d\vec{f}}{dt}$ exists, the limit will be a vector in the direction of the tangent to the curve $\vec{f}(t)$ at the point (f_1, f_2, f_3) and is given by

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{\imath} + \frac{df_2}{dt}\hat{\jmath} + \frac{df_3}{dt}\hat{k}$$

Here $\frac{df_1}{dt}, \frac{df_2}{dt}$, and $\frac{df_3}{dt}$ are the derivative of scalar function as

$$\frac{df_1}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f_1(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f_1(t + \Delta t) - f_1(t)}{\Delta t}$$

$$\frac{df_2}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f_2(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f_2(t+\Delta t)-f_2(t)}{\Delta t}$$

and $$\frac{df_3}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f_3(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f_3(t + \Delta t) - f_3(t)}{\Delta t}$$

Example: Find $\frac{d\vec{f}}{dt}$ if $\vec{f}(t) = (e^{2t} + 1)\hat{\imath} + \sin(2t)\hat{\jmath} + t^3\hat{k}$.

Solution: $\vec{f}(t) = (e^{2t} + 1)\hat{\imath} + \sin(2t)\hat{\jmath} + t^3\hat{k}$

By differentiating w.r.t t we get

$$\frac{d\vec{f}(t)}{dt} = \frac{d}{dt}\left[(e^{2t} + 1)\hat{\imath} + \sin(2t)\hat{\jmath} + t^3\hat{k}\right]$$

$$= \frac{d}{dt}(e^{2t} + 1)\hat{\imath} + \frac{d}{dt}[\sin(2t)]\hat{\jmath} + \frac{d}{dt}[t^3]\hat{k}$$

$$\frac{d\vec{f}}{dt} = 2e^{2t}\hat{\imath} + 2\cos(2t)\hat{\jmath} + 3t^2\hat{k}$$

5.4 Vector Differentiation

5.4.1 Prove the following formulae of differentiation

- $\frac{d\vec{a}}{dt} = 0$
- $\frac{d}{dt}\left[\vec{f} \pm \vec{g}\right] = \frac{d\vec{f}}{dt} \pm \frac{d\vec{g}}{dt}$
- $\frac{d}{dt}\left[\emptyset\vec{f}\right] = \emptyset\frac{d\vec{f}}{dt} + \frac{d\emptyset}{dt}\vec{f}$
- $\frac{d}{dt}\left[\vec{f}.\vec{g}\right] = \vec{f}\frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt}\vec{g}$
- $\frac{d}{dt}\left[\vec{f} \times \vec{g}\right] = \vec{f} \times \frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt} \times \vec{g}$
- $\frac{d}{dt}\left[\frac{f}{\emptyset}\right] = \frac{1}{\emptyset^2}\left[\emptyset\frac{d\vec{f}}{dt} - \vec{f}\frac{d\emptyset}{dt}\right]$

Where *a* is a constant vector function, *f* and *g* are vector functions, and $\emptyset$ is a scalar function of t.

In general, the standard rules of differentiation can also be extended to a vector function:

(i) $\frac{d\vec{a}}{dt} = 0$

Consider, $\vec{f}(t) = \vec{a}$ is a constant vector function

$$= a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$$

$$\frac{d\vec{f}}{dt} = \lim_{\Delta t\to 0}\left[\frac{(a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}) - (a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k})}{\Delta t}\right]$$

$$\frac{da}{dt} = \lim_{\Delta t\to 0}\left[\frac{a_1 - a_1}{\Delta t}\right]\hat{\imath} + \lim_{\Delta t\to 0}\left[\frac{a_2 - a_2}{\Delta t}\right]\hat{\jmath} + \lim_{\Delta t\to 0}\left[\frac{a_3 - a_3}{\Delta t}\right]\hat{k}$$

$$= \lim_{\Delta t\to 0}[0]\hat{\imath} + \lim_{\Delta t\to 0}[0]\hat{\jmath} + \lim_{\Delta t\to 0}[0]\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0$$

Hence, $\frac{da}{dt} = 0$

(ii) $\frac{d}{dt}[\vec{f} + \vec{g}] = \frac{d\vec{f}}{dt} + \frac{d\vec{g}}{dt}$

By definition

$$\frac{d}{dt}[\vec{f} + \vec{g}] = \lim_{\Delta t\to 0}\left[\frac{[(\vec{f} + \Delta\vec{f}) + (\vec{g} + \Delta\vec{g})] - (\vec{f} + \vec{g})}{\Delta t}\right]$$

$$\frac{d}{dt}[\vec{f} + \vec{g}] = \lim_{\Delta t\to 0}\left[\frac{[\vec{f} + \Delta\vec{f} + \vec{g} + \Delta\vec{g}] - (\vec{f} + \vec{g})}{\Delta t}\right] = \lim_{\Delta t\to 0}\left[\frac{[\vec{f} + \Delta\vec{f} + \vec{g} + \Delta\vec{g} - \vec{f} - \vec{g}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\left(\frac{(\vec{f} + \Delta\vec{f}) - \vec{f}}{\Delta t}\right) + \left(\frac{(\vec{g} + \Delta\vec{g}) - \vec{g}}{\Delta t}\right)\right] = \lim_{\Delta t\to 0}\frac{(\vec{f} + \Delta\vec{f}) - \vec{f}}{\Delta t} + \lim_{\Delta t\to 0}\frac{(\vec{g} + \Delta\vec{g}) - \vec{g}}{\Delta t}$$

$$\frac{d}{dt}[\vec{f} + \vec{g}] = \frac{d\vec{f}}{dt} + \frac{d\vec{g}}{dt}$$ Hence proved.

Similarly, $\frac{d[\vec{f} - \vec{g}]}{dt} = \frac{d\vec{f}}{dt} - \frac{d\vec{g}}{dt}$

(iii) $\frac{d}{dt}[\phi\vec{f}] = \phi\frac{d\vec{f}}{dt} + \vec{f}\frac{d\phi}{dt}$ **where ϕ is scalar function**

$$\frac{d[\phi\vec{f}]}{dt} = \lim_{\Delta t\to 0}\left[\frac{(\phi + \Delta\phi)(\vec{f} + \Delta\vec{f}) - \phi\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\phi + \Delta\phi)(\vec{f} + \Delta\vec{f}) + \phi(\vec{f} + \Delta\vec{f}) - \phi(\vec{f} + \Delta\vec{f}) - \phi\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f} + \Delta\vec{f})[(\phi + \Delta\phi) - \phi] + \phi[(\vec{f} + \Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)-\phi]}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{\phi[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)-\phi]}{\Delta t}\right]+\phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)-\phi]}{\Delta t}\right]+\phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \vec{f}\frac{d\phi}{dt}+\phi\frac{d\vec{f}}{dt}=\phi\frac{d\vec{f}}{dt}+\frac{d\phi}{dt}\vec{f}$$

Hence proved. $(\because\ \Delta t\to 0\ \ \therefore \Delta\vec{f}\to 0)$

(iv) $\boldsymbol{\frac{d}{dt}(\vec{f}\cdot\vec{g})=\vec{f}\cdot\frac{d}{dt}(\vec{g})+\vec{g}\cdot\frac{d}{dt}(\vec{f})}$

$$\frac{d(\vec{f}\cdot\vec{g})}{dt}=\lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\cdot(\vec{g}+\Delta\vec{g})+\vec{f}\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\vec{f}\cdot[(\vec{g}+\Delta\vec{g})-\vec{g}]+(\vec{g}+\Delta\vec{g})\cdot[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \vec{f}\cdot\lim_{\Delta t\to 0}\left[\frac{[(\vec{g}+\Delta\vec{g})-\vec{g}]]}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]\cdot\lim_{\Delta t\to 0}(\vec{g}+\Delta\vec{g})$$

$$= \vec{f}\cdot\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\cdot\vec{g}$$ proved

(v) $\boldsymbol{\frac{d}{dt}(\vec{f}\times\vec{g})=\vec{f}\times\frac{d\vec{g}}{dt}+\vec{g}\times\frac{d\vec{f}}{dt}}$

$$\frac{d(\vec{f}\times\vec{g})}{dt}=\lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\times(\vec{g}+\Delta\vec{g})-\vec{f}\times\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\times(\vec{g}+\Delta\vec{g})+\vec{f}\times(\vec{g}+\Delta\vec{g})-\vec{f}\times(\vec{g}+\Delta\vec{g})-\vec{f}\times\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\vec{f}\times[(\vec{g}+\Delta\vec{g})-\vec{g}]+[(\vec{f}+\Delta\vec{f})-\vec{f}]\times(\vec{g}+\Delta\vec{g})}{\Delta t}\right]$$

$$= \vec{f}\times\lim_{\Delta t\to 0}\left[\frac{[(\vec{g}+\Delta\vec{g})-\vec{g}]]}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]\times\lim_{\Delta t\to 0}(\vec{g}+\Delta\vec{g})$$

$$= \vec{f}\times\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\times\vec{g}$$ Hence proved. $(\because\ \Delta t\to 0\ \ \therefore \Delta\vec{g}\to 0)$

(vi) $\dfrac{d}{dt}\left[\dfrac{\vec{f}}{\phi}\right] = \dfrac{\phi\dfrac{d\vec{f}}{dt} - \vec{f}\dfrac{d\phi}{dt}}{\phi^2}$ [ϕ is scalar function]

$$\frac{d}{dt}\left[\frac{1}{\phi}\vec{f}\right] = \frac{d}{dt}[\phi^{-1}\vec{f}] = \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)^{-1}(\vec{f}+\Delta\vec{f}) + \phi^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)^{-1} - \phi^{-1}] + \phi^{-1}[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)^{-1} - \phi^{-1}]}{\Delta t}\right] + \lim_{\Delta t\to 0}\left[\frac{\phi^{-1}[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)^{-1} - \phi^{-1}]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[\phi^{-1}(1+\frac{\Delta\phi}{\phi})^{-1} - \phi^{-1}]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{\left[\phi^{-1}[1 - \left(\frac{\Delta\phi}{\phi}\right) + \left(\frac{\Delta\phi}{\phi}\right)^2 - \cdots] - \phi^{-1}\right]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

[Neglecting the terms involving higher power of $\Delta\phi$]

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{\left[\frac{1}{\phi^2}[\phi - \Delta\phi] - \phi\right]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\frac{1}{\phi^2}\lim_{\Delta t\to 0}\frac{[[\phi - \Delta\phi] - \phi]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \frac{1}{\phi^2}\left\{\phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right] - \lim_{\Delta f\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\left[\frac{\phi - [\phi - \Delta\phi]}{\Delta t}\right]\right]\right\}$$

$\because \Delta t \to 0, \therefore \Delta\vec{f} \to 0,$ thus

$$= \frac{1}{\phi^2}\left[\phi\frac{d\vec{f}}{dt} - \frac{d\phi}{dt}\vec{f}\right] \quad \text{proved}$$

Example 1. If $\vec{u} = 2t\hat{\imath} - 5\hat{\jmath}$; $\vec{v} = t^2\hat{\imath} - 2t\hat{k}$ are vector functions and $\emptyset(t) = 3t$ is scalar function then find the following:

(i) $\dfrac{d}{dt}[\vec{u}(t) + \vec{v}(t)]$ (ii) $\dfrac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)]$

(iii) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)]$ (iv) $\frac{d}{dt}[\emptyset\,\vec{u}(t)]$

Solution:

(i) $\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]+\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]$

$= (2\hat{\imath}) + (2t\hat{\imath}-2\hat{k}) = (2t+2)\hat{\imath}-2\hat{k}$

(ii) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}(t)\cdot\left[\frac{d}{dt}\vec{v}(t)\right]+\vec{v}(t)\cdot\left[\frac{d}{dt}\vec{u}(t)\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\cdot\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]+(t^2\hat{\imath}-2t\hat{k})\cdot\left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]$

$= (2t\vec{\imath}-5\vec{\jmath})\cdot(2t\hat{\imath}-2\hat{k})+(t^2\hat{\imath}-2t\hat{k})\cdot(2\hat{\imath})$ $\because[\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1]$

$= (4t^2)+(2t^2) = 6t^2$ $[\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{k}=\hat{k}\cdot\hat{\imath}=0]$

(iii) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}(t)\times\left[\frac{d}{dt}\vec{v}(t)\right]+\vec{v}(t)\times\left[\frac{d}{dt}\vec{u}(t)\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\times\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]+(t^2\hat{\imath}-2t\hat{k})\times\left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\times[2t\,\hat{\imath}-2\,\hat{k}]+(t^2\hat{\imath}-2t\hat{k})\times[2\hat{\imath}]$

$$= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2t & -5 & 0\\ 2t & 0 & -2\end{vmatrix}+\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ t^2 & 0 & -2t\\ 2 & 0 & 0\end{vmatrix}$$

$= \{[(-5)(-2)-0]\hat{\imath}-[(2t)(-2)-0]\hat{\jmath}+[0-(-5)(2t)]\,\hat{k}\}+$
$\{[0-0]\hat{\imath}-[0-(-2t)(2)]\hat{\jmath}+[0-0]\,\hat{k}\}$

$= \{[10]\hat{\imath}-[-4t]\hat{\jmath}+[10t]\hat{k}\}+\{-4t]\hat{\jmath}\} = \{[10]\hat{\imath}+[4t]\hat{\jmath}+[10t]\hat{k}\}-\{4t]\hat{\jmath}\}$

$= 10\vec{\imath}+10t\vec{k}$

(iv) $\frac{d}{dt}[\emptyset(t)\,\vec{u}(t)] = \emptyset(t)\frac{d}{dt}\vec{u}(t)+\vec{u}(t)\frac{d\emptyset}{dt}$

$= (3t)\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})+(2t\hat{\imath}-5\hat{\jmath})\frac{d}{dt}(3t)$

$= (3t)(2\hat{\imath}-0)+(2t\hat{\imath}+5\hat{\jmath})(3)$

$= 6t\hat{\imath}+6t\hat{\imath}+15\hat{\jmath}$

$= 12t\hat{\imath}+15\hat{\jmath}$

5.4.2 Apply vector differentiation to calculate velocity and acceleration of a position vector $\vec{r}(t) = x(t)\hat{\imath}+y(t)\hat{\jmath}+z(t)\hat{k}$

Consider $\vec{r}(t) = x(t)\,\hat{\imath}+y(t)\,\hat{\jmath}+z(t)\hat{k}$ is a position vector joining the origin O of the coordinate system at any point (x, y, z) as shown in the figure 5.3.

As t changes, the terminal point $\vec{r}(t)$ describe a curve having parametric equations

$$x = x(t), \qquad y = y(t), \; z = z(t)$$

If $\lim\limits_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$ exists then the rate of change $\frac{d\vec{r}}{dt}$ will be the velocity $\vec{v}$. We further differentiate velocity $\vec{v}$ with respect to time, we have $\frac{d\vec{v}}{dt}$ i.e., $\frac{d^2\vec{r}}{dt^2}$ which represents acceleration along the curve.

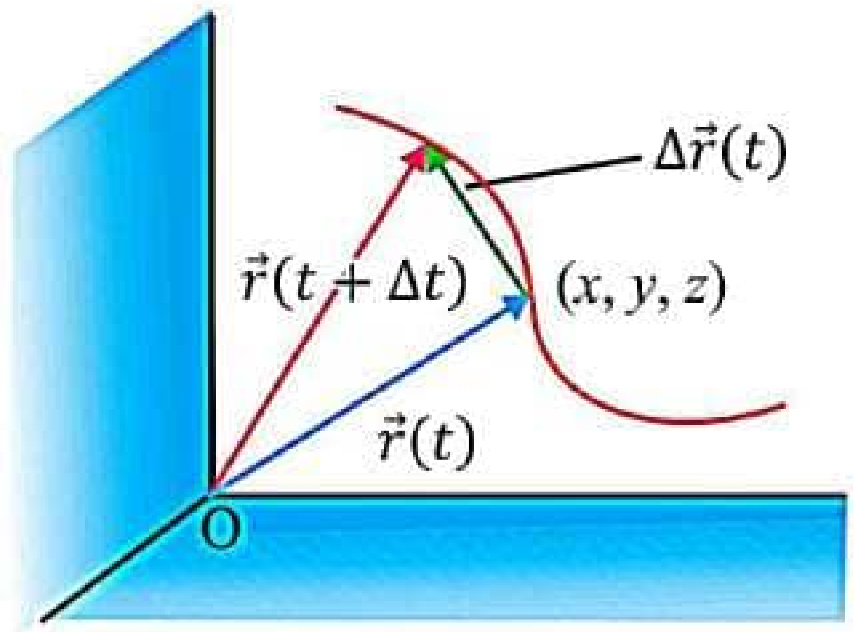


Fig. 5.3

Example 1. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t \quad z = 2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

Solution:

(a) The position vector of the particle is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\hat{k} = e^{-t}\,\hat{\imath} + 2\cos 3t\;\hat{\jmath} + 2\sin 3t\;\hat{k}$$

The velocity is $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[e^{-t}\,\hat{\imath} + 2\cos 3t\;\hat{\jmath} + 2\sin 3t\;\hat{k}\right]$

$$= \frac{d}{dt}(e^{-t})\,\hat{\imath} + 2\frac{d}{dt}(\cos 3t)\;\hat{\jmath} + 2\;\frac{d}{dt}(\sin 3t)\;\hat{k}$$

$$\vec{v} = -\,e^{-t}\hat{\imath} - 6\sin 3t\,\hat{\jmath} + 6\cos 3t\,\hat{k}$$

The acceleration is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[-\,e^{-t}\hat{\imath} - 6\sin 3t\,\hat{\jmath} + 6\cos 3t\,\hat{k}\right]$

$$= \left[\frac{d}{dt}[-\,e^{-t}]\,\hat{\imath} - 6\frac{d}{dt}[\cos 3t]\;\hat{\jmath} + 6\;\frac{d}{dt}[\cos 3t]\;\hat{k}\right]$$

$$\vec{a} = e^{-t}\vec{\imath} - 18\cos 3t\,\vec{\jmath} - 18\sin 3t\,\vec{k}$$

(b) At $t = 0$, the velocity is $\vec{v} = -\,e^{-(0)}\hat{\imath} - 6\sin 3(0)\,\hat{\jmath} + 6\cos 3(0)\,\hat{k}$

$$\vec{v} = -\hat{\imath} + 6\hat{k}$$

The magnitude of $\vec{v}$ i.e., $|\vec{v}| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$ units

At $t = 0$, the acceleration is $\vec{a} = e^{-(0)}\hat{\imath} - 18\cos 3(0)\,\hat{\jmath} - 18\sin 3(0)\,\hat{k}$

$$\vec{a} = \hat{\imath} - 18\hat{\jmath}$$

The magnitude of $\vec{a}$ i.e., $|\vec{a}| = \sqrt{(1)^2 + (18)^2} = \sqrt{325}$ units

Example 2. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$.

Solution:

(a) The position vector of the particle is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\hat{k} = 2t^2\,\hat{\imath} + (t^2 - 4t)\,\hat{\jmath} + (3t - 5)\,\hat{k}$$

The velocity is $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[2t^2\,\hat{\imath} + (t^2 - 4t)\,\hat{\jmath} + (3t - 5)\,\hat{k}\right]$

$$= \frac{d}{dt}[2t^2]\,\hat{\imath} + \frac{d}{dt}(t^2 - 4t)\,\hat{\jmath} + \frac{d}{dt}(3t - 5)\,\hat{k}$$

$$\vec{v} = 4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}$$

The acceleration is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}\right]$

$$= \left[\frac{d}{dt}[4t]\hat{\imath} + \frac{d}{dt}(2t - 4)\,\hat{\jmath} + \frac{d}{dt}[3]\hat{k}\right]$$

$$\vec{a} = 4\,\hat{\imath} + 2\,\hat{\jmath}$$

(b) At $t = 1$, the velocity is $\vec{v} = 4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}$

$$\vec{v} = 4\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}$$

At $t = 1$, the acceleration is $\vec{a} = 4\,\hat{\imath} + 2\,\hat{\jmath}$

The component of $\vec{v}$ along the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ is

$$\frac{(4\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}).(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \frac{(4)(1) + (-2)(-3) + (3)(2)}{\sqrt{1 + 9 + 4}} = \frac{16}{\sqrt{14}}$$

The component of $\vec{a}$ along the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ is

$$\frac{(4\,\hat{\imath} + 2\,\hat{\jmath}).(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \frac{(4)(1) + (2)(-3) + (0)(2)}{\sqrt{1 + 9 + 4}} = \frac{-2}{\sqrt{14}}$$

Exercise 5.2

1. Find the derivative of the following vector functions.
   - (i) $\vec{f}(t) = \ln t^2\,\hat{\imath} + e^{2t}\hat{\jmath} + (2t^2 + 1)\hat{k}$
   - (ii) $\vec{f}(t) = (t + 1)\hat{\imath} + \ln(t + 2)\,\hat{\jmath}$
   - (iii) $\vec{f}(t) = \sec t\,\hat{\imath} + \cos t^2\hat{\jmath} + (t^2 + t + 1)\hat{k}$
2. If $\vec{x} = t\hat{\imath} + 2t\,\hat{\jmath}$; $\vec{y} = 2t\hat{\imath} + 3t\hat{k}$ are vector functions and $\emptyset(t) = 3t$ is scalar function, then find the following:
   - (i) $\frac{d}{dt}\left[\vec{x}(t) - \vec{y}(t)\right]$ (ii) $\frac{d}{dt}\left[\vec{x}(t) \cdot \vec{y}(t)\right]$
   - (iii) $\frac{d}{dt}\left[\vec{x}(t) \times \vec{y}(t)\right]$ (iv) $\frac{d}{dt}\left[\emptyset\,\vec{x}(t)\right]$
3. A particle moves so that its position as a function of time is given by $\vec{r}(t) = \hat{\imath} + 4t^2\hat{\jmath} + t\,\hat{k}$. Write expressions for its:
   (a) velocity (b) acceleration as functions of time.
4. The path of a particle is given for time $t > 0$ by the parametric equations $x = t^2 - 3t$ and $y = \frac{2}{3}t^3$. Find magnitude of velocity and acceleration of particle at $t = 5$.

5. A particle moves along the curve $x = 2t^2$ and $y = 4t$. Find the component of velocity and acceleration at $t = 2$ in the direction of $2\hat{\imath} + \hat{\jmath}$.

Review Exercise 5

1. Choose the correct answer.

(i) The domain of the vector function $\vec{r}(t) = t^3\hat{\imath} + \frac{1}{t-1}\hat{\jmath} + \ln(t-2)\hat{k}$, $t \in R$

(a) $\{t > 2,, t \in R\}$ (b) $\{t < 2,, t \in R\}$

(c) $\{t > 0,, t \in R\}$ (d) $\{t > 1,, t \in R\}$

(ii) Is the Function $\vec{F}(t) = \cos t\,\hat{\imath} + \sin 2t\ \hat{k}$ at $t = \pi$ ----------------

(a) is not continuous (b) is continuous (c) limit does not exist (d) None of these

(iii) What value of t, vector function $\vec{F}(t) = \frac{1}{t-2}\hat{\imath} + \sin t\ \hat{\jmath}$ is continuous at

(a) $t \neq 2,\ t \in R$ (b) $t > 2,\ t \in R$ (c) $t \leq 2,\ t \in R$ (d) $t \in R$

(iv) If $\vec{u} = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ is a vector function and $h(t) = t^2 + 2t - 2$ is scalar function then $\lim_{t\to 0} [h(t)\,\vec{u}(t)] =$ --------------

(a) $-\hat{\jmath}$ (b) $\hat{\jmath}$ (c) $2\hat{\jmath}$ (d) $-2\hat{\jmath}$

(v) If $\vec{u} = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ and $\vec{v} = -2\hat{\imath} - t^2\hat{k}$ are vector functions then $\lim_{t\to 0} [\vec{u}(t) \cdot \vec{v}(t)] =$ ---------------

(a) 0 (b) t^4 (c) $t^2 - e^{2t}$ (d) $-2\,t^2$

(vi) If $\vec{f}(t) = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ then $\vec{f}'(t) =$ ----------------

(a) $2t\hat{\imath} - 2e^{2t}\hat{\jmath}$ (b) $2t\hat{\imath} - e^{2t}\hat{\jmath}$ (c) $2t\hat{\imath} - 2te^{2t}\hat{\jmath}$ (d) None of these

(vii) The velocity function of a particle, whose motion is given by $\vec{r}(t) = \frac{1}{2\pi}t\,\hat{\imath} + 3\cos(t)\,\hat{\jmath}$ at time $t = \frac{\pi}{2}$ is

(a) $\frac{1}{2\pi}\hat{\imath} + 3\hat{\jmath}$ (b) $\frac{1}{2\pi}\hat{\imath}$ (c) $\frac{1}{2\pi}\hat{\imath} - 3\hat{\jmath}$ (d) None of these

(viii) The magnitude of velocity of a particle at $t = \pi$, whose motion is given by $\vec{r}(t) = 4\cos(t)\,\hat{\imath} + 4\sin(t)\,\hat{\jmath} + \frac{3}{2\pi}t^2\hat{k}$

(a) $\sqrt{5}$ (b) 5 (c) $\sqrt{5\pi}$ (d) 5π

(ix) The magnitude of acceleration of a particle at $t = \frac{\pi}{2}$, whose motion is given by $\vec{r}(t) = 4\cos(t)\,\hat{\imath} + 4\sin(t)\,\hat{\jmath} + \frac{3}{2\pi}t^2\hat{k}$ is:

(a) $\frac{\sqrt{16\pi^2+9}}{\pi}$ (b) $\sqrt{16\pi^2+9}$ (c) $\frac{\sqrt{16\pi^2+9}}{\pi^2}$ (d) $\frac{\sqrt{16+9\pi^2}}{\pi}$

(x) Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ then $\dfrac{d\vec{a}}{dt}$ =------------

(a) 10 (b) $\sqrt{38}$ (c) 0 (d) None of these

2. Find the limit of vector function $\vec{r}(t) = 2t\,\hat{\imath} + t^3\,\hat{\jmath} + \hat{k}$ when $t \to 2$.
3. If $\vec{u} = 5\hat{\imath} - 2t\hat{\jmath}$; $\vec{v} = \hat{\imath} - 3t\hat{k}$ are vector functions and $k = t + 1$ is scalar function then find the following:

(i) $\lim\limits_{t\to 0} [\vec{u}(t) - \vec{v}(t)]$ (ii) $\lim\limits_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$

(iii) $\lim\limits_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim\limits_{t\to 5} [k\,\vec{u}(t)]$

4. Check the continuity of the function $\vec{G}(t) = (t+8)\,\hat{\imath} + \frac{6}{t-8}\,\hat{\jmath} + \ln\,(t+8)\,\hat{k}$ at $t = 0$.
5. For what value of t, following vector functions are continuous

$\vec{r}(t) = \sqrt{36 - t^2}\hat{\imath} + \ln\,(t+4)\,\hat{\jmath}$

6. Find $\vec{f}'(t)$ of the following vector functions.

(i) $\vec{f}(t) = e^{t^4}\hat{\imath} + (t^3 + 3)\hat{\jmath} + \text{cosec}\, t^2\,\hat{k}$

(ii) $\vec{f}(t) = \frac{1}{t}\hat{\imath} + e^{2t^3}\hat{\jmath} + \sec t^3\,\hat{k}$

7. A particle moves along the curve whose parametric equations are $x = t^3 + 2t$, $y = 3e^{-2t}$, $z = 2\sin(5\,t)$, where x, y and z show variations of the distance covered by the particle in cm with time in seconds. Find the magnitude of the acceleration of the particle at $t = 0$.
8. The path of a particle is given for time $t > 0$ by the parametric equations $x = t + \frac{2}{t}$ and $y = 3t^2$. Find velocity of particle when time $t = 1$ and acceleration at $t = 2$.

Integration

Unit 6

6.1 Introduction

Integration is the reverse process of differentiation. It is used in dealing with problems in which the derivative of a function, or its rate of change is known and we want to find the function.

The principles of integration were formulated independently by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century.

Integration is usually used to find area under a curve and volume of solid of revolution.

6.1.1 Demonstrate the concept of integral as an accumulator

Integral is the outcome of the process of integration and is of two types definite and indefinite. It is an accumulator which is used to find the definite integral of a function $f(x)$, which is continuous on a closed interval $[a, b]$. In this process, the region bounded by the geometrical curve of function $f(x)$, x-axis and the vertical lines $x = a$ and $x = b$ is divided into infinitesimal vertical rectangles each of width Δx on x-axis and height $f(x_i)$ from x-axis, where $i = 1, 2, 3, \ldots, n$ as shown in figure 6.1. The accumulation of the product $f(x_i)\Delta x$ is approximately equal to the definite integral of $f(x)$ from $x = a$ to $x = b$.

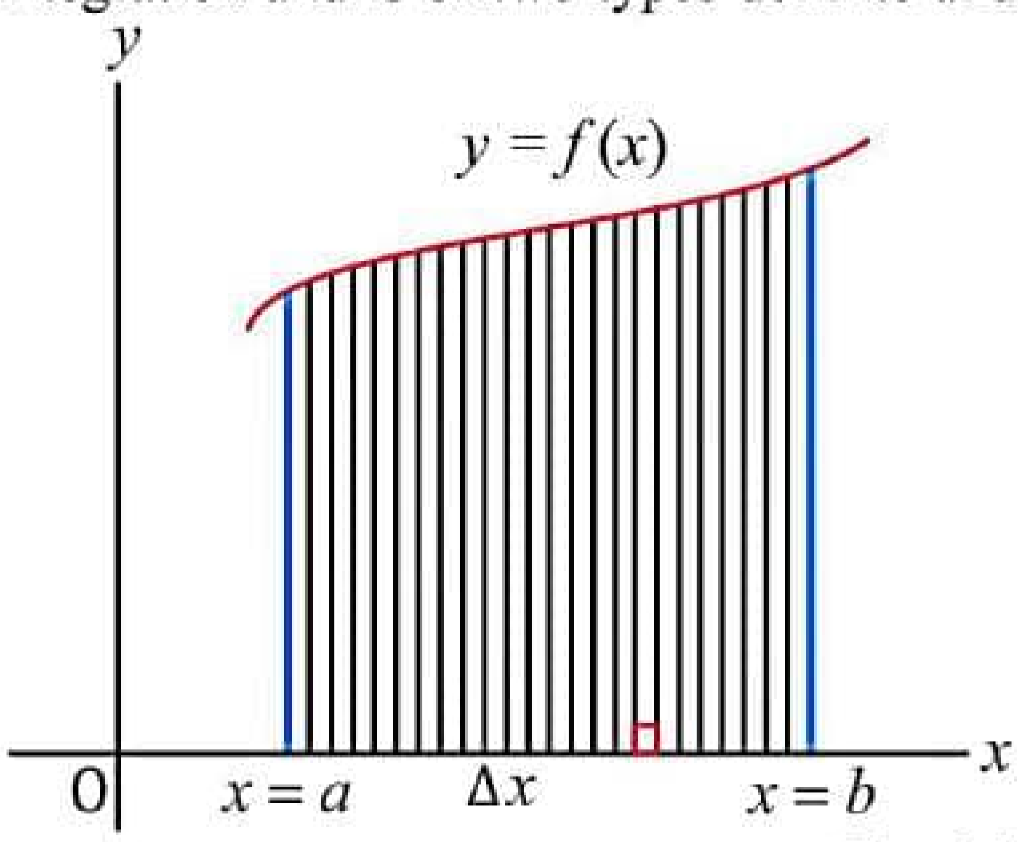


Fig. 6.1

Definite integral of $f(x)$ from a to b is Accumulation of $f(x_i)\Delta x$ as shown in the Fig. 6.1.

6.1.2 Know integration as inverse process of differentiation

In differentiation we are given an original function and we are required to find its derivative, while in integration we are required to find the original function whose derivative is given.

$f(x)$ —differentiation→ $f'(x)$
Original function → derivative / derived function

$f'(x)$ —integration→ $f(x)$
Derived function → original function

Thus, integration is the reverse or inverse process of differentiation.

Indefinite integral or antiderivative

Let f be a continuous function. A function F whose derivative is f is called integral of f, i.e., $F'(x) = f(x)$, $\forall x$ in the domain of $f(x)$.

As F is an integral or antiderivative or primitive of a function f, then

$$\int f(x)dx = F(x) + C$$

where $f(x)$ is called integrand and C is called the constant of integration. The solution $F(x) + C$, depends on the arbitrary constant C, so $\int f(x)dx$ has indefinite solution, called indefinite integral.

6.1.3 Explain Constant of integration

Since, the derivative of a constant is 0, therefore all the functions which differ by constant have the same derivative.

For example:

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$$
$$f(x) = x^3 \pm 1 \quad \Rightarrow \quad f'(x) = 3x^2$$
$$f(x) = x^3 \pm c \quad \Rightarrow \quad f'(x) = 3x^2$$

So, it is not possible to find original function through integration, therefore, we use a constant C in indefinite integral to represent the family of functions whose derivatives are the function $f(x)$. This arbitrary constant C is called constant of integration.

Note: In $\int f(x)\,dx = F(x) + C$,

- $\int$ is the integral sign (elongated S) which is used to represent the process of integration.
- $f(x)$ is the integrand; a function which is to be integrated or under the effect of integral sign.
- dx, x is the variable of integration that tells integrand is to be integrated w.r.t x.
- C is the integral constant or constant of integration.
- $F(x) + C$ represents family of integrals or antiderivatives or primitives whose derivatives are $f(x)$.

6.1.4 Know simple standard integrals which directly follow from standard differentiation formulae.

The basic integration formulae can be obtained directly from the differentiation formulae of the functions as given I the following table.

S. No.	Differentiation formulae	Integration formulae
1.	$\frac{d}{dx}(x+c)=1$	$\int dx = x + C$
2.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right)=x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1}+C, \quad n \neq -1$
3.	$\frac{d}{dx}\left\{\frac{(ax+b)^{n+1}}{a(n+1)}+c\right\}=(ax+b)^n$	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C,$ $n \neq -1$
4.	$\frac{d}{dx}(\ln\|x\|+c)=\frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$

Trigonometric Functions

5.	$\frac{d}{dx}(\sin x + c) = \cos x$	$\int \cos x\, dx = \sin x + C$
6.	$\frac{d}{dx}(\cos x + c) = -\sin x$	$\int \sin x\, dx = -\cos x + C$
7.	$\frac{d}{dx}(\ln\|\sec x\| + c) = \tan x$	$\int \tan x\, dx = \ln\|\sec x\| + C$
8.	$\frac{d}{dx}(\ln\|\sin x\| + c) = \cot x$	$\int \cot x\, dx = \ln\|\sin x\| + C$
9.	$\frac{d}{dx}[\ln\|\sec x + \tan x\| + c] = \sec x$	$\int \sec x\, dx = \ln\|\sec x + \tan x\| + C$
10.	$\frac{d}{dx}[\ln\|\text{cosec}\, x - \cot x\| + c] = \text{cosec}\, x$	$\int \text{cosec}\, x\, dx = \ln\|\text{cosec}\, x - \cot x\| + C$
11.	$\frac{d}{dx}(\tan x + c) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
12.	$\frac{d}{dx}(\cot x + c) = -\text{cosec}^2 x$	$\int \text{cosec}^2 x\, dx = -\cot x + C$
13.	$\frac{d}{dx}(\sec x + c) = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
14.	$\frac{d}{dx}(\text{cosec}\, x + c) = -\text{cosec}\, x \cot x$	$\int \csc x \cot x\, dx = -\text{cosec}\, x + C$

Inverse Trigonometric Functions

15.	$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}+c\right)=\frac{1}{\sqrt{a^2-x^2}}$	$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C$
16.	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}+c\right)=\frac{1}{a^2+x^2}$	$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
17.	$\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}+c\right)=\frac{1}{x\sqrt{x^2-a^2}}$	$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}+C$

Exponential Functions

18.	$\frac{d}{dx}(e^x+c)=e^x$	$\int e^x dx=e^x+C$
19.	$\frac{d}{dx}\left(\frac{1}{a}e^{ax}+c\right)=e^{ax}$	$\int e^{ax}dx=\frac{1}{a}e^{ax}+C$
20.	$\frac{d}{dx}\left(\frac{a^x}{\ln a}+c\right)=a^x$	$\int a^x dx=\frac{a^x}{\ln a}+C$

Some other Integration Formulae

21.	$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+C$
22.	$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+C$
23.	$\int\frac{dx}{\sqrt{x^2+a^2}}=\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert+C$
24.	$\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert+C$
25.	$\int\sqrt{a^2-x^2}dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\text{Sin}^{-1}\frac{x}{a}+C$
26.	$\int\sqrt{a^2+x^2}dx=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\ln\left\lvert x+\sqrt{a^2+x^2}\right\rvert+C$
27.	$\int\sqrt{x^2-a^2}dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert+C$

6.2 Rules of integration

6.2.1 Recognize the following rules of integration

(i) $\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\int f(x)dx = f(x) + C$ **where C is the constant**

As we know that integration is an inverse process of differentiation. So, the process of differentiation and integration neutralizes each other.

i.e., $$\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\int f(x)\,dx = f(x) + C$$

Example: $$\int \left[\frac{d}{dx}(x^2 + 5x + 7)\right] dx = x^2 + 5x + 7 + C$$

(ii) **The integral of the product of a constant and a function is the product of the constant and the integral of the function.**

Let k be a constant and $f(x)$ be a function then,

$$\int k\,f(x)dx = k\int f(x)dx$$

Example: $$\int 4e^x\,dx = 4\int e^x dx$$
$$= 4\,e^x + c$$

(iii) **The integral of the sum of a finite number of functions is equal to the sum of their integrals.**

Let $f(x)$, $g(x)$ and $h(x)$ are three differentiable functions whose integrals exist then.

$$\int [f(x) + g(x) + h(x)]\,dx = \int f(x)dx + \int g(x)dx + \int h(x)dx$$

Example:

$$\int (9x^8 + 6x^5 + 5)dx = \int 9x^8\,dx + \int 6x^5\,dx + \int 5\,dx$$
$$= 9\int x^8\,dx + 6\int x^5\,dx + 5\int dx$$
$$= 9\left(\frac{x^9}{9}\right) + 6\left(\frac{x^6}{6}\right) + 5(x) + C$$
$$= x^9 + x^6 + 5x + C$$

6.2.2 Use standard differentiation formulae to prove the results for the following integrals

(i) $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \qquad n \neq -1$

Proof: Consider the differentiation

$$\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}+C\right\}=\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}\right\}+\frac{d}{dx}(C)$$

$$=\frac{1}{n+1}\frac{d}{dx}[f(x)]^{n+1}+0$$

$$=\frac{1}{n+1}(n+1)[f(x)]^{n+1-1}\frac{d}{dx}f(x)$$

$$=[f(x)]^{n}f'(x)$$

By taking indefinite integral on both sides

$$\int\left[\frac{d}{dx}\left\{\frac{[f(x)]^{n+1}}{n+1}+C\right\}\right]dx=\int[f(x)]^{n}f'(x)dx$$

$$\frac{[f(x)]^{n+1}}{n+1}+C=\int[f(x)]^{n}f'(x)dx$$

$$\boxed{\int[f(x)]^{n}f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+C,n\neq-1}$$

(ii) $\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+C$

Proof: Consider the differentiation

$$\frac{d}{dx}(\ln|f(x)|+c)=\frac{d}{dx}\ln|f(x)|+\frac{d}{dx}(C)\qquad\left(\because\frac{d}{dx}\ln|x|=\frac{1}{x}\right)$$

$$=\frac{1}{f(x)}\frac{d}{dx}f(x)+0$$

$$=\frac{f'(x)}{f(x)}$$

By taking integrals on both sides

$$\int\frac{d}{dx}(\ln f(x)+c)dx=\int\frac{f'(x)}{f(x)}dx$$

$$\ln f(x)+c=\int\frac{f'(x)}{f(x)}dx$$

$$\boxed{\int\frac{f'(x)}{f(x)}dx=\ln f(x)+C}$$

(iii) $\int e^{ax}[af(x)+f'(x)]dx$

Proof: Consider the differentiation

$$\frac{d}{dx}[e^{ax}f(x)+c]=\frac{d}{dx}[e^{ax}f(x)]+\frac{d}{dx}(c)$$

By using product rule of differentiation

$$=f(x)\frac{d}{dx}e^{ax}+e^{ax}\frac{d}{dx}f(x)+0$$

$$=f(x)(ae^{ax})+e^{ax}f'(x)$$

$$=e^{ax}[af(x)+f'(x)]$$

By taking integrals on both sides

$$\int\frac{d}{dx}[e^{ax}f(x)+c]dx=\int e^{ax}[af(x)+f'(x)]dx$$

$$\boxed{\int e^{ax}[af(x)+f'(x)]dx=e^{ax}f(x)+C}$$

when $a=1$

$$\boxed{\int e^{x}[f(x)+f'(x)]dx=e^{x}f(x)+C}$$

Example: Evaluate

(i) $\int(10x^2+5)^8(20x)dx$

Solution:

Let $f(x)=10x^2+5$

$f'(x)=20x$

By the rule of integration

$$\int(10x^2+5)^8(20x)dx=\frac{(10x^2+5)^9}{9}+C$$

(ii) $\int\frac{2x+3}{x^2+3x-5}dx$

Solution: Let $f(x)=x^2+3x-5, f'(x)=2x+3$

By the rule of integration

$$\int\frac{2x+3}{x^2+3x-5}dx=\ln|x^2+3x-5|+C$$

(iii) $\int e^{x}(2x^2+4x)dx$

Let $f(x)=2x^2, f'(x)=4x$

By the rule of integration

$$\int e^{x}(2x^2+4x)dx=2e^{x}x^2+C$$

Exercise 6.1

1. Evaluate the following indefinite integrals by using standard formulae.

(i) $\int 9x^5\,dx$ (ii) $\int \frac{15}{x^3}\,dx$

(iii) $\int \frac{a}{\sqrt{bx}}\,dx$ (iv) $\int by^{\frac{2}{3}}dy$

(v) $\int(3x^2 - 9x + 5)\,dx$ (vi) $\int(2x^{-5} + 3x^{-2})\,dx$

(vii) $\int \frac{(x^5+3x^3-5x+6)}{x^4}dx$ (viii) $\int(\cos x + 3\sin x)\,dx$

(ix) $\int(3\sec x - \operatorname{cosec} x)\,dx$ (x) $\int(2\tan x - 5\sec x)\,dx$

(xi) $\int(9e^x - 3\cos x - 5\sin x)\,dx$ (xii) $\int(\sec^2 x + \operatorname{cosec}^2 x)\,dx$

2. Evaluate the following indefinite integrals by using standard formulae.

(i) $\int(3x^2 + 9x + 3)^{\frac{1}{2}}(6x + 9)dx$ (ii) $\int \sqrt{ax^2 + 2bx + c}\,(ax + b)dx$

(iii) $\int \frac{(6x+5)dx}{\sqrt{3x^2+5x+2}}dx$ (iv) $\int(x^2 + 4x + 3)^{-9}(2x + 4)dx$

(v) $\int(x - 2)(x - 3)(x - 4)dx$ (vi) $\int(2x^2 - 3)^2dx$

(vii) $\int(x^3 - 3x^2 + 9)^{\frac{7}{2}}(x^2 - 2x)dx$ (viii) $\int(x^2 - 5)^3dx$

(ix) $\int(cos\,x + \sin x)^{\frac{3}{2}}(\cos x - \sin x)dx$

(x) $\int(\tan x + \sin x)(\sec^2 x + \cos x)dx$

3. Evaluate by using standard formulae of integration.

(i) $\int \frac{x}{x^2+3}dx$ (ii) $\int \frac{\sec^2 x + \cos x}{\tan x + \sin x}dx$

(iii) $\int \frac{5x^4+4x^3-3x^2+2x}{x^5+x^4-x^3+x^2}dx$ (iv) $\int \frac{\left(e^x+\frac{1}{x}\right)dx}{e^x+\ln x}$

(v) $\int(5x^3 - 3x^2 + 6x - 9)^{-1}(5x^2 - 2x + 2)dx$ (vi) $\int \frac{1}{x+\sqrt{x}}dx$

4. Evaluate the following integrals by using standard formulae.

(i) $\int e^x(\sin x + \cos x)dx$ (ii) $\int e^x\left(\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right)dx$

(iii) $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$ (iv) $\int e^x(\sec^2 x + \tan x)dx$

(v) $\int e^x\left(\frac{1}{x} + \ln x\right)dx$

5. Evaluate the following integrals by using appropriate formulae.

(i) $\int \frac{dx}{x^2+9}$ (ii) $\int \frac{dt}{\sqrt{4-t^2}}$ (iii) $\int \frac{dy}{y\sqrt{y^2-9}}$

(iv) $\int \frac{dt}{4t^2-9}$ (v) $\int \frac{dx}{\sqrt{9x^2+16}}$ (vi) $\int \frac{dx}{\sqrt{16x^2-9}}$

(vii) $\int \frac{dx}{9-x^2}$ (viii) $\int \frac{dx}{x\sqrt{4x^2-16}}$ (ix) $\int \sqrt{9-4x^2}\, dx$

(x) $\int \sqrt{25+9x^2}\, dx$ (xi) $\int \frac{dy}{9y^2+81}$ (xii) $\int \frac{dx}{4x^2-16}$

6.3 Integration by substitution

6.3.1 Explain the method of integration by substitution

Sometimes the integrals of the form $\int f(g(x))g'(x)\, dx$ can be converted in standard form or easier by substituting $g(x)$ by introducing a new variable.

To understand the integration by substitution method.

Suppose $u = g(x)$

By the differentials $du = g'(x)dx$

Thus,

$$\int f(g(x))\, g'(x) = \int f(u)\, du$$

Obviously, the integral on the right is much easier to evaluate than that on the left.

6.3.2 Apply method of substitution to evaluate indefinite integrals

Example 1. Evaluate by substitution method.

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx$$

Splitting x^3, we get

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx = \int (2x^2-3)^{\frac{5}{2}} x^2 (xdx)$$

Now suppose $u = 2x^2 - 3 \implies x^2 = \frac{u+3}{2}$

$\implies du = 4xdx$

$\implies \frac{1}{4}du = xdx$

By substituting, we get

$$\int (2x^2-3)^{\frac{5}{2}} x^3 dx = \int u^{\frac{5}{2}} \left(\frac{u+3}{2}\right)\left(\frac{1}{4}du\right)$$

$$= \frac{1}{8}\int \left(u^{\frac{7}{2}} + 3u^{\frac{5}{2}}\right) du$$

$$= \frac{1}{8}\left[\frac{u^{\frac{9}{2}}}{\frac{9}{2}} + 3\frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right] + C$$

$$= \frac{1}{4}\left[\frac{u^{\frac{9}{2}}}{9} + 3\frac{u^{\frac{7}{2}}}{7}\right] + C$$

Replacing u by $2x^2 - 3$, we get

$$= \frac{1}{36}(2x^2 - 3)^{\frac{9}{2}} + \frac{3}{28}(2x^2 - 3)^{\frac{7}{2}} + C$$

Example 2. Evaluate $\int \cos^4 3x \sin 3x \, dx$

Solution: Substitute $u = \cos 3x$

$\Rightarrow$ $du = -3\sin 3x \, dx$

$$-\frac{1}{3}du = \sin 3x \, dx$$

By substituting

$$\int \cos^4 3x \sin 3x \, dx = \int u^4 \left(-\frac{1}{3}du\right)$$

$$= -\frac{1}{3}\left(\frac{u^5}{5}\right) + C$$

$$= -\frac{1}{15}\cos^5 3x + C$$

Example 3. Evaluate $\int \cos^2 x \sin^3 x \, dx$

Solution: $\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx$

$$= \int \cos^2 x \, (1 - cos^2 x) \sin x \, dx$$

Substitute $u = \cos x$

$\Rightarrow$ $du = -\sin x \, dx$

$$\int \cos^2 x \sin^3 x \, dx = -\int u^2(1 - u^2) \, du$$

$$= -\int (u^2 - u^4) \, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Example 4. Evaluate $\int \sin^2 2x \cos^2 2x \, dx$

Solution: By multiplying and dividing by 2^2.

$$\int \sin^2 2x \cos^2 2x \, dx = \frac{1}{(2)^2}\int (2)^2 \sin^2 2x \cos^2 2x$$

$$= \frac{1}{4}\int (2\sin 2x \cos 2x)^2 \, dx$$

$$= \frac{1}{4}\int \sin^2 4x \, dx$$

$$= \frac{1}{4}\int \left(\frac{1-\cos 8x}{2}\right) dx$$

$$= \frac{1}{8}\int (1-\cos 8x) \, dx$$

$$= \frac{1}{8}\left[x - \frac{\sin 8x}{8}\right] + C$$

$$= \frac{1}{8}x - \frac{1}{64}\sin 8x + C$$

Example 5. Evaluate $\int \sin 4x \cos 2x \, dx$

Solution: $\int \sin 4x \cos 2x \, dx = \int \frac{1}{2}[\sin(4x+2x) + \sin(4x-2x)\, dx]$

$$= \frac{1}{2}\int (\sin 6x + \sin 2x)\, dx \quad (\because \text{ Using trigonometric identities})$$

$$= \frac{1}{2}\left(-\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right) + C$$

$$= -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + C$$

Example 6. Evaluate $\int \tan^5 x \, dx$

Solution: $\int \tan^5 x \, dx = \int \tan^3 x \tan^2 x \, dx$

$$= \int \tan^3 x\,(\sec^2 - 1)dx$$

$$= \int (\tan^3 x \sec^2 x - \tan^3 x)\, dx$$

$$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

$$= \frac{\tan^4 x}{4} - \int \tan x \tan^2 x \, dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x\,(\sec^2 x - 1)\, dx$$

$$= \frac{1}{4}\tan^4 x - \int \tan x \sec^2 dx + \int \tan x \, dx$$

$$=\frac{1}{4}\tan^4 x-\frac{\tan^2 x}{2}+\ln|\sec x|+C$$
$$=\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+C$$

Example 7. Evaluate $\int \sec^6 x\,dx$

Solution:
$$\int \sec^6 x\,dx=\int \sec^4 x\sec^2 x\,dx$$
$$=\int (\sec^2 x)^2\sec^2 x\,dx$$
$$=\int (1+\tan^2 x)^2\sec^2 x\,dx$$

Substitute $u=\tan x$

$du=\sec^2 x\,dx$

By substituting

$$=\int (1+u^2)^2 du$$
$$=\int (1+2u^2+u^4)du$$
$$=u+\frac{2}{3}u^3+\frac{u^5}{5}+C$$
$$=\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C$$

Exercise 6.2

1. Evaluate the following integrals by substitution method.

(i) $\int \frac{3x}{\sqrt{x^2+7}}$ (ii) $\int x^{\frac{4}{3}}\left(a^{\frac{7}{3}}-x^{\frac{7}{3}}\right)^{\frac{5}{2}}dx$

(iii) $\int \frac{1+2x}{\sqrt{1-x}}dx$ (iv) $\int (2x^2+4x+5)^{\frac{3}{2}}(x+1)dx$

(v) $\int \frac{x}{\sqrt{1-x^2}}dx$ (vi) $\int (x^2+2x+5)^{-1}(x+1)dx$

(vii) $\int x^3(9+x^2)^{\frac{3}{2}}dx$ (viii) $\int (x^3-9)^{\frac{5}{2}}x^5dx$

(ix) $\int x^9(x^5+3)^{\frac{2}{5}}dx$ (x) $\int (x^3+x^2+5x-1)^{-1}(3x^2+2x+5)dx$

2. Evaluate the following integrals by substitution method.

(i) $\int \frac{\ln x}{x}dx$ (ii) $\int \frac{dx}{x\ln x}$ (iii) $\int \frac{\tan x}{\ln(\cos x)}dx$

(iv) $\int \frac{\tan(\ln x)}{x}dx$ (v) $\int (1+e^{2x})^{-\frac{1}{2}}e^{2x}dx$

(vi) $\int \frac{5e^{5\tan x}}{\cos^2 x} dx$ (vii) $\int \frac{e^{2x}}{e^x+e^{-x}} dx$

(viii) $\int e^{(\text{cosec}\, 2x+1)} \cdot (\text{cosec}\, 2x \cot 2x) dx$

(ix) $\int 3^x\, dx$ (x) $\int e^{(\sin x+\cos x+3)} \cdot (\cos x - \sin x) dx$

3. Evaluate the following integrals by substitution method.

(i) $\int \cos^5 2x \sin 2x\, dx$ (ii) $\int \frac{\cot\sqrt{x}}{\sqrt{x}} dx$

(iii) $\int (2+\sin 3x)^6 \cos 3x\, dx$ (iv) $\int \sin(ax+b)\, dx$

(v) $\int \frac{\sin x+\cos x}{\sin x-\cos x} dx$ (vi) $\int e^{2x} \sec^2 e^{2x} dx$

(vii) $\int \frac{\text{cosec}\, 3x \cot 3x dx}{(a+b\, \text{cosec}\, 3x)^2} dx$ (viii) $\int \frac{dx}{(2\cot x+3)\sin^2 x}$

(ix) $\int \frac{\sec x dx}{3\sin x+4\cos x}$ (x) $\int \cos(3x-5)\, dx$

4. Evaluate the following integrals by substitution method.

(i) $\int \cos^2 2y\, dy$ (ii) $\int \sin^3(3x+5) dx$

(iii) $\int \cos^3 x \sqrt{\sin x}\, dx$ (iv) $\int \sin^4 x \cos^5 x\, dx$

(v) $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$ (vi) $\int \cos^5 x \sin^7 x\, dx$

(vii) $\int \sin^3 x \cos^3 x\, dx$ (viii) $\int \sin 2x \cos 4x\, dx$

(ix) $\int \cos 3x \cos 5x\, dx$ (x) $\int \sin 3x \cos 7x\, dx$

(xi) $\int \tan^2 x\, dx$ (xii) $\int \cot^4 x\, dx$

(xiii) $\int \tan^7 x\, dx$ (xiv) $\int \sec^4 2x\, dx$

(xv) $\int \tan^5 3x \sec^3 3x\, dx$ (xvi) $\int \text{cosec}^4 3x\, dx$

(xvii) $\int \sec^4 x \sqrt{\tan x}\, dx$ (xviii) $\int \cot 2x\, \text{cosec}^4 2x\, dx$

(xix) $\int \frac{\text{cosec}^4 x}{\sqrt{\cot x}} dx$ (xx) $\int \sqrt{1+\cos x}\, dx$

6.3.3 Apply method of substitution to evaluate integrals of the following types:

$\int \frac{dx}{a^2-x^2}$	$\int \sqrt{a^2-x^2}\, dx$	$\int \frac{dx}{\sqrt{a^2-x^2}}$
$\int \frac{dx}{a^2+x^2}$	$\int \sqrt{a^2+x^2}\, dx$	$\int \frac{dx}{\sqrt{a^2+x^2}}$
$\int \frac{dx}{ax^2+bx+c}$	$\int \frac{dx}{\sqrt{ax^2+bx+c}}$	$\int \frac{px+q}{ax^2+bx+c} dx$
$\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$		

The above types of integrals can be solved easily by using trigonometric substitutions.

If the integrand contains expression of the form $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2}$; we use the following trigonometric substitutions:

S.No.	Integrand	Substitution	
1.	$\sqrt{a^2-x^2}$	$x = a\sin\theta$	or $x = a\cos\theta$
2.	$\sqrt{a^2+x^2}$	$x = a\tan\theta$	or $x = a\cot\theta$
3.	$\sqrt{x^2-a^2}$	$x = a\sec\theta$	or $x = a\,\text{cosec}\,\theta$

If the integral involves a quadratic expression in the denominator or under a radical, we first make the expression perfect square by using $a^2 \pm 2ab + b^2 = (a \pm b)^2$, then use suitable trigonometric substitution.

Example 1. Evaluate $\int \frac{dx}{4-x^2}$

(i) $\int \frac{dx}{4-x^2} = \int \frac{dx}{2^2-x^2}$

Substituting $x = 2\sin\theta$

$\Rightarrow \quad dx = 2\cos\theta\, d\theta$

$$\therefore \int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\, d\theta}{4-4\sin^2\theta}$$

$$= \int \frac{2\cos\theta\, d\theta}{4(1-\sin^2\theta)}$$

$$= \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\cos^2\theta}$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \quad \ldots\text{(i)}$$

Now,

$x = 2\sin\theta$

$\therefore \quad \sin\theta = \frac{x}{2}$

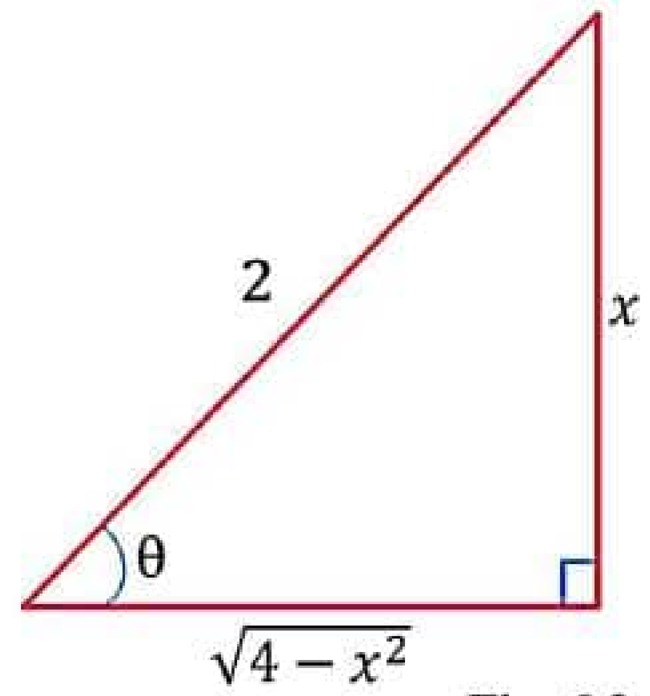


Fig. 6.2

From the figure 6.2, we have

$$\sec\theta = \frac{2}{\sqrt{4-x^2}}$$

$$\boxed{\tan\theta = \frac{x}{\sqrt{4-x^2}}}$$

By putting the values in equation (i)

$$\int \frac{dx}{4-x^2} = \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{4-x^2}}\right| + c$$

Thus, $\int \frac{dx}{4-x^2} = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + C$

$$\int \frac{dx}{4-x^2} = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$$

(ii) $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{dx}{\sqrt{3^2+x^2}}$

Substituting $x = 3\tan\theta$

$\Rightarrow \quad dx = 3\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9(1+\tan^2\theta)}}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

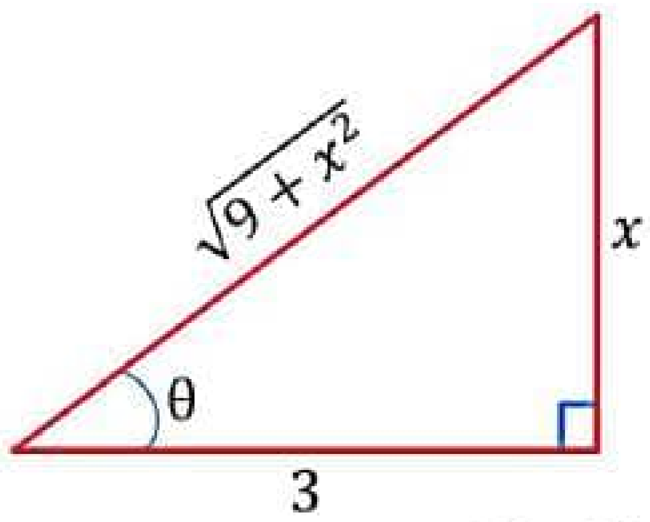


Fig. 6.3

As $x = 3\tan\theta \quad = \int \sec\theta\, d\theta$

$$= \ln|\sec\theta + \tan\theta| + C \qquad \ldots\text{(i)}$$

$\therefore \tan\theta = \frac{x}{3}$

From figure 6.3, we have

$$\sec\theta = \frac{\sqrt{9+x^2}}{3}$$

Thus, $\int \frac{dx}{\sqrt{9+x^2}} = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c_1$

$$= \ln\left|\frac{x+\sqrt{9+x^2}}{3}\right| + c_1$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + (c_1 - \ln 3)$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + C, \qquad \text{where } C = c_1 - \ln 3$$

$$= \ln\left|x+\sqrt{9+x^2}\right| + C$$

(iii) $\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{x^2-5^2}}{x}dx$

Substituting $x = 5\sec\theta$

$\Rightarrow \quad dx = 5\sec\theta\tan\theta\, d\theta$

$$\therefore \quad \int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta\, d\theta)$$

$$= \int \sqrt{25(\sec^2\theta - 1)}\tan\theta\, d\theta$$

$$= \int 5\tan\theta\,(\tan\theta)d\theta$$

$$= \int 5\tan^2\theta\, d\theta$$

$$= 5\int (\sec^2\theta - 1)\, d\theta \qquad \ldots\text{(i)}$$

From the Fig. 6.4, as $x = 5\sec\theta = 5(\tan\theta - \theta) + C$

$\therefore \quad \sec\theta = \frac{x}{5} \quad \Rightarrow \quad \theta = \sec^{-1}\frac{x}{5}$

$\therefore \quad \tan\theta = \frac{\sqrt{x^2-25}}{5}$

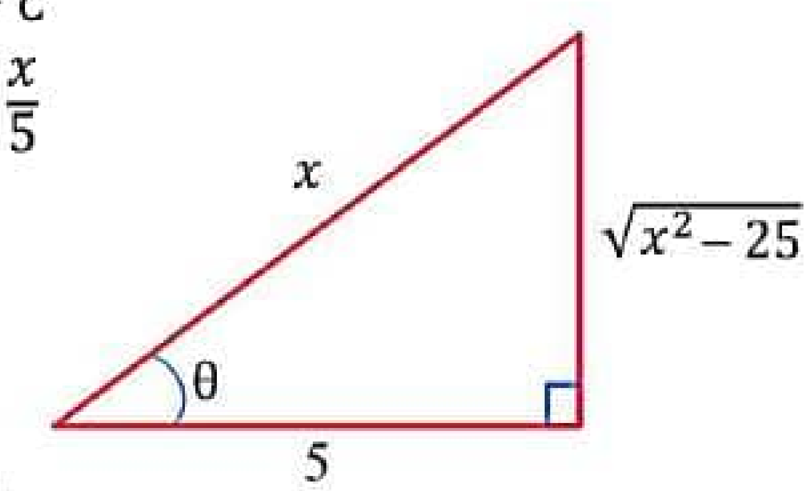


Fig. 6.4

From equation (i)

Thus, $\int \frac{\sqrt{x^2-25}}{5}dx = 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\frac{x}{5}\right) + C$

$$= \sqrt{x^2-25} - 5\sec^{-1}\frac{x}{5} + C$$

(iv) $\int \frac{dx}{x^2-4x+8}$

The expression in the denominator is quadratic, so before trigonometric substitution we make it completely square.

i.e., $x^2 - 4x + 8 = (x)^2 - 2(x)(2) + (2)^2 + 4$

$$= (x-2)^2 + 4$$

$$= (x-2)^2 + 2^2$$

$$\therefore \quad \int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2 + (2)^2}$$

Now, substituting $x - 2 = 2\tan\theta$

$\Rightarrow \quad dx = 2\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{x^2-4x+8} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta + 4}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4(\tan^2\theta + 1)}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\int d\theta$$

As $x - 2 = 2\tan\theta$ $\quad = \frac{1}{2}\theta + C$

$\therefore \quad \theta = \tan^{-1}\left(\frac{x-2}{2}\right)$

Thus $\quad \int \frac{dx}{x^2-4x+8} = \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$

(v) $\quad \int \frac{(2x+5)dx}{x^2+2x+5} = \int \frac{2x+2+3}{x^2+2x+5}dx$

$$= \int \frac{2x+2}{x^2+2x+5}dx + \int \frac{3}{x^2+2x+5}dx$$

$$= \ln|x^2+2x+5| + 3\int \frac{dx}{x^2+2x+1+4}$$

$$\int \frac{(2x+5)}{x^2+2x+5}dx = \ln|x^2+2x+5| + 3\int \frac{dx}{(x+1)^2+(2)^2} \qquad \ldots\text{(i)}$$

Let us find $\int \frac{dx}{(x+1)^2+2^2}$ by trigonometric substituting, we have

$x + 1 = 2\tan\theta$

$\Rightarrow \quad dx = 2\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{(x+1)^2+2^2} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta + 4}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4(\tan^2\theta + 1)}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + c$$

As $x + 1 = 2\tan\theta \quad \Rightarrow \quad \tan\theta = \frac{x+1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{x+1}{2}\right)$

$$\therefore \int \frac{dx}{(x+1)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Now, equation (i), becomes

$$\int \frac{(2x+5)dx}{x^2+2x+5} = \ln|x^2+2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Exercise 6.3

Evaluate by using trigonometric substitution.

1. $\int \frac{x^3 dx}{\sqrt{9-x^2}}$
2. $\int \frac{6dx}{9-x^2}$
3. $\int x^2\sqrt{9-x^2}\,dx$
4. $\int \frac{5dx}{25x^2+9}$
5. $\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$
6. $\int \frac{dx}{\sqrt{16+4x^2}}$
7. $\int x^3\sqrt{9x^2-36}$
8. $\int \frac{dx}{\sqrt{a^2+x^2}}$
9. $\int \frac{dx}{(16-x^2)^{\frac{5}{2}}}$
10. $\int \frac{x^5 dx}{\sqrt{x^2-9}}$
11. $\int \frac{dx}{x^2+4x+5}$
12. $\int \frac{dx}{\sqrt{5+4x-x^2}}$
13. $\int \frac{dx}{\sqrt{9x-x^2}}$
14. $\int \frac{dx}{(x+1)\sqrt{x^2+2x-15}}$
15. $\int \frac{dx}{(x-4)\sqrt{x^2-8x-9}}$
16. $\int \frac{(2x-5)dx}{\sqrt{8x-x^2}}$
17. $\int \frac{(x+3)dx}{x^2+2x+5}$
18. $\int \frac{(3x+9)dx}{x^2+4x+4}$
19. $\int \frac{(4x+9)dx}{\sqrt{2x^2+8x-10}}$
20. $\int \frac{(2x-5)dx}{\sqrt{5+4x-x^2}}$

6.4 Integration by Parts

6.4.1 Recognize the formula for integration by Parts

The method of integration by parts is used to integrate the product of two functions. Suppose that $f(x)$ and $g(x)$ are two functions and $f'(x)$ and $g'(x)$ are their derivatives respectively which exist in the domain of $f(x)$ and $g(x)$.

According to product rule of differentiation

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Integrating both sides w.r.t x, we get

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int \left[f(x)\frac{d}{dx}g(x)\right]dx + \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

$$f(x)g(x) = \int \left[f(x)\frac{d}{dx}g(x)\right]dx + \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

$$\int \left[f(x)\frac{d}{dx}g(x)\right]dx = f(x)g(x) - \int \left[g(x)\frac{d}{dx}f(x)\right]dx$$

Suppose $u = f(x)$

$v = \frac{d}{dx}g(x) \quad \Rightarrow \quad \int v dx = g(x)$

$$\boxed{\int uv\,dx = u\int vdx - \int \left(u'\int v\,dx\right)\,dx}$$

This result is called the formula for integration by parts. It can be stated as integral of the product of two functions equals first function same into integral of second function minus integral of product of derivative of 1st function and integral of 2nd function.

6.4.2 Apply method of integration by parts to evaluate integrals of the following types

- $\int \sqrt{a^2 - x^2}\,dx, \int \sqrt{a^2 + x^2}\,dx, \int \sqrt{x^2 - a^2}\,dx$

Example 1. Evaluate

(i) $\int \sqrt{a^2 - x^2}\,dx$

Let $\quad I = \int \sqrt{a^2 - x^2}\,dx = \int \sqrt{a^2 - x^2}\,(1)dx$

Integration by parts

$$I = \sqrt{a^2 - x^2}\int (1)\,dx - \int \left[\frac{d}{dx}(a^2 - x^2)^{\frac{1}{2}}\int (1)\,dx\right]dx$$

$$I = \sqrt{a^2 - x^2}(x) - \int \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)(x)dx$$

$$I = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}dx$$

$$I = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}dx$$

$$I = x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}}\right)dx$$

$$I = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$I = x\sqrt{a^2 - x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right) + C_1$$

$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C_1$$

$$I = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example 2. Evaluate $\int \sqrt{a^2 + x^2}\, dx$.

Solution: Let $I = \int \sqrt{a^2 + x^2}\, dx$

Integration by parts

$$I = \sqrt{a^2 + x^2}\int (1)dx - \int \left[\frac{d}{dx}\sqrt{a^2 + x^2}\int (1)dx\right] dx$$

$$I = x\sqrt{a^2 + x^2} - \int \frac{2x^2}{2\sqrt{a^2 + x^2}}\, dx$$

$$I = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$I = x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$I = x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2}{\sqrt{a^2 + x^2}} + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$I = x\sqrt{a^2 + x^2} - \int \sqrt{x^2 + a^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$2I = x\sqrt{a^2 + x^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$I = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

Example 3. Evaluate $\int \sqrt{x^2 - a^2}\, dx$.

Solution: Let $I = \int \sqrt{x^2 - a^2}\, dx$

Integration by parts

$$I = \sqrt{x^2 - a^2}\int (1)dx - \int \left[\frac{d}{dx}\sqrt{x^2 - a^2}\int (1)dx\right] dx$$

$$I = \sqrt{x^2 - a^2}(x) - \int \frac{2x^2}{2\sqrt{x^2 - a^2}}\, dx$$

$$I = x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} - \int \frac{a^2}{\sqrt{x^2 - a^2}}dx$$

$$I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}dx - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$2I = x\sqrt{x^2 - a^2} - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

6.4.3 Evaluate integrals using integration by parts

Example 1.

(i) $\int x \sin x \, dx$

Solution: Let $u = x$, $v = \sin x$

Integrating by parts, we have

$$\int x \sin x \, dx = x \int \sin x \, dx - \int \left(\frac{d}{dx}(x) \int \sin x\right) dx$$

$$= x \int (-\cos x) - \int (1)\,(-\cos x) dx$$

$$= -x \cos x + \sin x + C$$

or $$= \sin x - x \cos x + C$$

(ii) $\int x^3 \ln x \, dx$

Solution:

Let $u = \ln x$ and $v = x^3$

Integrating by parts, we have

$$\int uv \, dx = u \int v dx - \int \left(u' \int v dx\right) dx$$

$$\therefore \quad \int x^3 \ln x \, dx = \ln x \int x^3 \, dx - \int \left(\frac{d}{dx}\ln x \int x^3\right) dx$$

$$= \frac{x^4}{4}\ln x - \int \frac{1}{x}\left(\frac{x^4}{4}\right) dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$$

(iii) $\int x\cot^{-1}x\,dx$

Solution:

Let $u=\cot^{-1}x$ and $v=x$

$$\int x\cot^{-1}x\,dx=\int(u\,v)\,dx$$

Integrating by parts, we get

$$\int x\cot^{-1}x\,dx=u\int v dx-\int\left(\frac{du}{dx}\int v dx\right)dx$$

$$=\cot^{-1}x\int x\,dx-\int\left(\frac{d}{dx}\cot^{-1}x\int x\,dx\right)dx$$

$$=\frac{x^2}{2}\cot^{-1}x-\int\frac{-1}{1-x^2}\left(\frac{x^2}{2}\right)dx$$

$$=\frac{1}{2}x^2\cot^{-1}x+\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$

$$=\frac{1}{2}x^2\cot^{-1}x+\frac{1}{2}\int\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{1}{2}x^2\cot^{-1}x+\frac{1}{2}\int dx-\frac{1}{2}\int\frac{dx}{1+x^2}$$

$$\int x\cot^{-1}x\,dx=\frac{1}{2}x^2\cot^{-1}x+\frac{1}{2}x-\frac{1}{2}\tan^{-1}x+C$$

(iv) $\int e^x\sin x\,dx$

Solution: Let $\mathrm{I}=\int e^x\sin x\,dx$

selecting $u=\sin x$ and $v=e^x$

Integration by parts, we have

$$\int uv\,dx=u\int v\,dx-\int\left(u'\int v\,dx\right)dx$$

$$\mathrm{I}=\sin x\int e^x\,dx-\int\left(\frac{d}{dx}\sin x\int e^x\,dx\right)dx$$

$$\mathrm{I}=e^x\sin x-\int e^x\cos x\,dx$$

Re-integrating by parts, we get

$$\mathrm{I}=e^x\sin x-\left(e^x\cos x-\int e^x(-\sin x)\,dx\right)$$

$$\Rightarrow\quad \mathrm{I}=e^x\sin x-e^x\cos x-\int e^x\sin x\,dx$$

$\Rightarrow \quad I + I = e^x(\sin x - \cos x) + C_1$

$\Rightarrow \quad 2I = e^x(\sin x - \cos x) + C_1$

$\Rightarrow \quad \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C, \quad \text{where } C = \frac{C_1}{2}$

Exercise 6.4

1. Integrate by parts the following:

(i) $\int x^2 e^x dx$ (ii) $\int x^3 e^x dx$ (iii) $\int x \cos x\, dx$

(iv) $\int \ln x\, dx$ (v) $\int x^2 \sin x\, dx$ (vi) $\int x\, \text{cosec}^2 x\, dx$

(vii) $\int x \sec^2 x\, dx$ (viii) $\int (\ln x)^2\, dx$

2. Integrate by parts the following:

(i) $\int (x+1)\ln(x+1)\, dx$ (ii) $\int x \ln x\, dx$

(iii) $\int x^{-3} \ln x\, dx$ (iv) $\int x^{-4} \ln x^2\, dx$

(v) $\int \sin x \cos x \ln(\sin x)\, dx$ (vi) $\int \frac{\tan x}{\cos^2 x} \ln(\tan x)\, dx$

(vii) $\int \cot x\, cosec^2 x \ln(\cot x)\, dx$ (viii) $\int \sec^3 x \tan x \ln(\sec x)\, dx$

(ix) $\int \text{cosec}\, x \cot x \ln(\text{cosec}\, x)\, dx$ (x) $\int \frac{\ln x^2}{x^2} dx$

(xi) $\int \sec^3 x\, dx$ (xii) $\int \text{cosec}^3 x\, dx$

3. Integrate by parts the following:

(i) $\int 3x \cos(3x)\, dx$ (ii) $\int x^2 \sin x\, dx$

(iii) $\int \frac{x}{\cot^2 x} dx$ (iv) $\int x \sec^2 x\, dx$

(v) $\int \frac{5x}{\sin^2 2x} dx$ (vi) $\int \cos\sqrt{x}\, dx$

(vii) $\int e^{2x} \sin 2x\, dx$ (viii) $\int e^{-x} \cos 2x\, dx$

(ix) $\int \cos(\ln x)\, dx$ (x) $\int e^{ax} \sin bx\, dx$

4. Integrate by parts the following:

(i) $\int \sin^{-1} 3x\, dx$ (ii) $\int x^4 \tan^{-1} x\, dx$

(iii) $\int \tan^{-1}(2x) dx$ (iv) $\int x \cos^{-1} x\, dx$

(v) $\int 3x^2 \sin^{-1}(3x)\, dx$ (vi) $\int 2x \sec^{-1} x\, dx$

(vii) $\int 6x \operatorname{cosec}^{-1}(2x) dx$ (viii) $\int x^2 \cot^{-1} x\, dx$

5. Integrate by parts the following:

(i) $\int \sqrt{9-x^2}\, dx$ (ii) $\int \sqrt{16+4x^2}\, dx$ (iii) $\int \sqrt{x^2-25}\, dx$

6.5 Integration using partial fractions

6.5.1 Use partial fraction to find $\int \frac{f(x)}{g(x)} dx$, where $f(x)$ and $g(x)$ are polynomial functions such that $g(x) \neq 0$.

Example 1. Evaluate by using partial fraction

(i) $\int \frac{2x-5}{x^2-5x+6} dx$

Solution: $\int \frac{2x-5}{x^2-5x+6} dx$

By factorization $= \frac{2x-5}{x^2-5x+6} = \frac{2x-5}{(x-2)(x-3)}$

Let $\frac{2x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ ...(i)

$\Rightarrow \frac{2x-5}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\Rightarrow 2x-5 = A(x-3)+B(x-2)$ ...(ii)

As (ii) is an identity, so putting $x = 3$, we get

$\boxed{B = 1}$

Similarly, putting $x = 2$ in (ii), we get

$\boxed{A = 1}$

Identity (i), becomes

$$\frac{2x-5}{(x-2)(x-3)} = \frac{1}{x-2} + \frac{1}{x-3}$$

Integrating on both sides w.r.t x, we get

$$\int \frac{2x-5}{x^2-5x+6} dx = \int \frac{1}{x-2} dx + \int \frac{1}{x-3} dx$$

$= \ln|x-2| + \ln|x-3| + C_1$

$= \ln|x-2| + \ln|x-3| + \ln|C|$ where $C_1 = \ln|C|$

$= \ln|C\,(x-2)(x-3)|$

(ii) $\int \frac{\cos x dx}{\sin x(2+\sin x)}$

Solution: To express the integrand in polynomial.

Suppose $u = \sin x$

$du = \cos x\, dx$

$$\int \frac{\cos x\, dx}{\sin x\,(2+\sin x)} = \int \frac{du}{u(2+u)} \qquad \ldots(i)$$

Let

$$\frac{1}{u(2+u)} \equiv \frac{A}{u} + \frac{B}{2+u}$$

$$1 = A(2+u) + Bu$$

$$\Rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\frac{1}{u(2+u)} = \frac{1}{2u} - \frac{1}{2(2+u)}$$

By putting in equation (i)

$$\int \frac{\cos x\, dx}{\sin x\,(2+\sin x)} = \int \left\{\frac{1}{2u} - \frac{1}{2(2+u)}\right\} du$$

$$= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|2+u| + \ln|C|$$

$$= \ln\left|u^{\frac{1}{2}}\right| - \ln\left|2+u^{\frac{1}{2}}\right| + \ln|C|$$

$$= \ln\left|C\frac{\sqrt{u}}{\sqrt{2+u}}\right|$$

$$= \ln\left|C\frac{\sqrt{\sin x}}{\sqrt{2+\sin x}}\right|$$

(iii) $\int \frac{(5x^2+1)dx}{(x-1)(x+2)^2} dx$

Solution: Partial fraction

$$\frac{5x^2+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$5x^2 + 1 \equiv A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

To solve it we get $\quad A = \frac{2}{3}, B = \frac{19}{21}, c = -\frac{1}{7}$

$$\frac{5x^2+1}{(x-1)(x+2)^2} = \frac{2}{3(x-1)} + \frac{19}{20(x+2)} - \frac{1}{7(x+2)^2}$$

Thus,

$$\int \frac{(5x^2+1)dx}{(x-1)(x+2)^2} = \frac{2}{3}\int \frac{dx}{x-1} + \frac{19}{20}\int \frac{dx}{x+2} - \frac{1}{7}\int \frac{dx}{(x+2)^2}$$

$$= \frac{2}{3}\ln|x-1| + \frac{19}{20}\ln|x+2| + \frac{1}{7(x+2)} + c$$

(iv) $\int \frac{x+1}{x(x^2+2)} dx$

Solution: Let $\frac{x+1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$ ...(i)

or $\frac{x+1}{x(x^2+2)} = \frac{A(x^2+2)+x(Bx+C)}{x(x^2+2)}$

$\Rightarrow$ $x+1 = A(x^2+2) + x(Bx+C)$ ...(ii)

As (ii) is an identity, so putting $x = 0$, we get

$$A = \frac{1}{2}$$

$$B - C = -\frac{3}{2}$$

$$B + C = \frac{1}{2}$$

By solving, we get

$B = -\frac{1}{2}$ and $C = 1$

Now, identity (i), becomes

$$\frac{x+1}{x(x^2+2)} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}x+1}{x^2+2}$$

Integrating both sides w.r.t x, we get

$$\int \frac{x+1}{x(x^2+2)} dx = \frac{1}{2}\int \frac{1}{x} dx - \frac{1}{2}\int \frac{x}{x^2+2} dx + \int \frac{1}{x^2+2} dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Exercise 6.5

Evaluate the following integrates by using partial fraction.

1. $\int \frac{(5x-2)dx}{(x-3)(x+7)}$

2. $\int \frac{(7x-25)dx}{(x-3)(x-4)}$

3. $\int \frac{dx}{a^2-x^2}$

4. $\int \frac{dx}{x^2-a^2}$

5. $\int \frac{(x^2+2x+3)dx}{x^3-x}$

6. $\int \frac{5dx}{x^2-2x-15}$

7. $\int \frac{(2x+7)dx}{(x-1)(x-5)(x+3)}$

8. $\int \frac{(5x+6)dx}{(x+3)(x-2)^2}$

9. $\int \frac{(7x^2-2x+5)dx}{(x-6)(x-3)^3}$

10. $\int \frac{(2x+1)dx}{(x-3)(x^2+1)}$

11. $\int \frac{sec^2 xdx}{(1+tan\,x)(2+tan\,x)}$

12. $\int \frac{cosec^2 xdx}{cot\,x(2+cot\,x)}$

13. $\int \frac{(3x+7)dx}{(2x-1)(x-4)^2}$

14. $\int \frac{(7x-4)dx}{(x-3)(x^2+2)}$

15. $\int \frac{(2x^2+5x+1)}{x^2+5x+6}dx$

16. $\int \frac{(x^3+3x+1)dx}{x^2+5x-14}$

6.6 Definite integrals

6.6.1 Define definite integral as the limit of a sum.

Suppose that $f(x)$ is a continuous function on the interval $[a, b]$, divide the interval $[a, b]$ into n infinitesimal sub intervals as

$$a = x_0 \leq x_1 \leq x_2 \leq x, \dots \dots \leq x_{n-1} \leq x_n = b$$

If Δx be the width of each subinterval, then

$\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, 3, \dots \dots, n$ as $\Delta x_i \to 0$ and $n \to \infty$

Select a point c_i on each interval such that

$$x_{i-1} \leq c_i \leq x_i$$

The limit of the sum

$$= \lim_{n\to\infty} \{f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n\}$$

By using summation notation it can be written a

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This summation of $f(x)$ on infinitesimal sub intervals is defined as the definite integral of $f(x)$ from a to b denoted by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Where a and b are called lower and upper limits of the integral respectively.

6.2.2 Describe fundamental theorem of integral calculus and recognize the following basic properties:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$
- $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

Fundamental theorem of integral calculus:

If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$

i.e., $\frac{d}{dx}F(x) = f(x)$ then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

is called fundamental theorem of calculus.

Example: $\int_0^3 (x^2 + 5)dx.$

Solution: Let $f(x) = x^2 + 5$ then its antiderivative $F(x) = \frac{x^3}{3} + 5x$

Now, by using fundamental theorem of calculus

$$\int_0^3 (x^2 + 5)\,dx = F(3) - F(0)$$

$$= \frac{(3)^3}{3} + 5(3) - \frac{(0)^3}{3} - 5(0)$$

$$= 9 + 15$$

$$= 24$$

Basic properties of definite integrals

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad a < c < b$

(v) $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

(i) $\int_a^a f(x)\,dx = 0$

Proof: By the fundamental theorem of integral calculus

$$\int_a^a f(x)\,dx = F(a) - F(a)$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

Proof: By the fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

and

$$\int_a^b f(y)\,dy = F(b) - F(a)$$

Hence

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: By the fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = -\{F(a) - F(b)\}$$

Thus,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad a < c < b$

Proof: By the fundamental theorem of integral calculus

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= \int_a^b f(x)\,dx$$

Thus,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(v) $\int_{-a}^{a} f(x)\,dx = \begin{cases} \int_0^a f(x)\,dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$

Proof: By using the property (iv), we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\,, \text{ where } -a < 0 < a \qquad \ldots\text{(i)}$$

Suppose $x = -t \quad \Rightarrow \quad dx = -dt$

When $x = -a \ \Rightarrow t = a$

When $x = 0 \ \Rightarrow t = 0$

By substituting in (i), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-t)\,(-dt) + \int_{0}^{a} f(x)dx$$

$$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)dx$$

By using property (iii), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)dx \qquad \ldots\text{(ii)}$$

When $f(-x) = f(x)$

From equation (ii)

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$$

When $f(-x) = -f(x)$

From equation (ii), we have

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$

6.6.3 Extend techniques of integration using properties to evaluate definite integrals

Example 1. Evaluate $\int_2^2 5x^4 dx$.

Solution: As the upper and lower limits are equal, by using property of $\int_a^a f(x)\,dx = 0$.

Hence

$$\int_2^2 5x^4\,dx = 0$$

Example 2. Shown that $\int_1^2 x^2\,dx = \int_1^2 y^2\,dy$.

Solution:

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]^2$$

$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}$$

By the property $\int_a^b f(x)\,dy = \int_a^b f(y)\,dy$

Thus,

$$\int_1^2 y^3\,dx = \frac{7}{3}$$

Hence

$$\int_1^2 x^2\,dx = \int_1^2 y^2\,dy$$

Example 3. Verify that $\int_0^{\frac{\pi}{2}} \cos x\,dx = -\int_{\frac{\pi}{2}}^0 \cos x\,dx$.

Solution:

$$\int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

Now $$-\int_{\frac{\pi}{2}}^0 \cos x\,dx = -[\sin x]_{\frac{\pi}{2}}^0$$

$$= -\left(\sin 0 - \sin\frac{\pi}{2}\right)$$

$$= 1$$

Hence

$$\boxed{\int_0^{\frac{\pi}{2}} \cos x\,dx = -\int_{\frac{\pi}{2}}^0 \cos x\,dx}$$ verified.

Example 4. Given that $\int_0^1 f(x)\,dx = 5$ and $\int_1^4 f(x)\,dx = 3$, then evaluate $\int_4^0 f(x)\,dx$.

Solution:

As $\int_4^0 f(x)\,dx = -\int_0^4 f(x)\,dx$ (By using property iii)

$= -\left[\int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx\right]$ (By using property iv)

$= -(5+3)$

$$\int_4^0 f(x)\,dx = -8$$

Example 5. Evaluate $\int_{-2}^1 |x|\,dx$

Solution: By property (iv), we have

$$\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx \quad \because \begin{cases} |x| = +x, x > 0 \\ |x| = -x, x < 0 \end{cases}$$

$$= \int_{-2}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$= -\frac{1}{2}[0-4] + \frac{1}{2}[1-0]$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

Example 6. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

Solution: Here $f(x) = \cos x$

$f(-x) = \cos(-x)$

$f(-x) = \cos x$

$f(-x) = f(x)$

By using property $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_{0}^{\frac{\pi}{2}} \cos x\,dx$$

$$= 2[\sin x]_0^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(1 - 0)$$

$$= 2$$

Example 7. Evaluate $\int_{-1}^{1} \sin^{-1} x\, dx$

Solution: Here $f(x) = \sin^{-1} x$

$$f(-x) = \sin^{-1}(-x)$$

$$f(-x) = -\sin^{-1} x$$

$$f(-x) = -f(x)$$

∴ f is an odd function.

Furthermore, limits are additive inverses of each other.

∴ By property (v), we have

$$\int_{-1}^{1} \sin^{-1} x\, dx = 0$$

Example 8. Evaluate $\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}}$ by using trigonometric substitution.

Solution: Let $x = 3\sec\theta \implies dx = 3\sec\theta\tan\theta\; d\theta$

When $x = -2\sqrt{3} \implies -2\sqrt{3} = 3\sec\theta \implies \sec\theta = -\frac{2}{\sqrt{3}} \implies \theta = \frac{5\pi}{6}$

When $x = -6 \implies -6 = 3\sec\theta \implies \sec\theta = -2 \implies \theta = \frac{2\pi}{3}$

$$\therefore \int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2-9}} = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{3\sec\theta\tan\theta\; d\theta}{3\sec\theta\sqrt{(3\sec\theta)^2 - 9}}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta\; d\theta}{\sqrt{9(\sec^2\theta - 1)}}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta\; d\theta}{3\sqrt{\tan^2\theta}} \qquad \left(\because \sqrt{x^2} = |x|\right)$$

$$\therefore \quad = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta \; d\theta}{3|\tan\theta|}$$

As $\theta \in \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right] \implies \tan\theta < 0 \implies |\tan\theta| = -\tan\theta$

$$\therefore \quad = \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \frac{\tan\theta \; d\theta}{3(-\tan\theta)}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} -\frac{1}{3} d\theta = \left[-\frac{1}{3}\theta\right]_{\frac{2\pi}{3}}^{\frac{5\pi}{6}}$$

$$= -\frac{1}{3}\left[\frac{5\pi}{6} - \frac{2\pi}{3}\right]$$

$$= -\frac{\pi}{18}$$

Example 9. Evaluate $\int_0^{\frac{\pi}{4}} x \sin x \, dx$.

Solution: Integrating by parts

$$\int_0^{\frac{\pi}{4}} x \sin x \, dx = \left[x \int \sin x \, dx\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left(\frac{dx}{dx} \int \sin x \, dx\right)$$

$$\int_0^{\frac{\pi}{4}} x \sin x \, dx = [-x \cos x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x \, dx$$

$$= -\left[\frac{\pi}{4}\cos\frac{\pi}{4} - 0\right] + \left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= -\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - 0\right)$$

$$= \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{-\pi + 4}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8}(4 - \pi)$$

Exercise 6.6

1. Evaluate the following definite integrals.

(i) $\int_0^2 (4x^3 + 3x^2 + 5)\, dx$ (ii) $\int_{-1}^2 (x^2 + 1)^2\, dx$

(iii) $\int_1^2 \left(\theta + \sqrt{\theta}\right)^3 d\theta$ (iv) $\int_1^3 \left(y + \frac{1}{\sqrt{y}}\right)^2 dy$

(v) $\int_0^4 \frac{dx}{\sqrt{2+x}+\sqrt{x}}$

2. Evaluate the following definite integrals or by formula.

(i) $\int_1^4 x(x^2 + 9)^{\frac{3}{2}}\, dx$ (ii) $\int_2^5 \frac{x dx}{7x^2+2}$

(iii) $\int_0^3 \frac{(2x+3)dx}{\sqrt{2x^2+6x+5}}$ (iv) $\int_1^2 (x^3 + 2x)^{-\frac{1}{2}} (3x^2 + 2)dx$

(v) $\int_0^{\frac{\pi}{2}} \cos^3 x\, dx$ (vi) $\int_{\frac{\pi^2}{4}}^{\frac{\pi^2}{36}} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

(vii) $\int_0^{\frac{\pi}{4}} \sqrt{\tan x} \sec^2 x\, dx$ (viii) $\int_0^2 e^{5x-2}\, dx$

(ix) $\int_0^{\frac{\pi}{3}} \cos^3 3x \sin^2 3x\, dx$ (x) $\int_0^{\frac{\pi}{3}} \tan^2 x\, sec^4 x\, dx$

3. Compute the following definite integrals by using basic properties.

(i) $\int_2^2 (x^4 + 2x + 3)^{\frac{5}{2}} (2x^3 + 1)dx$

(ii) $\int_{-50}^{50} (10x^9 - 8x^7 + 6x^5 - 4x^3 + 2x)\, dx$

(iii) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^9 x \cos^6 x\, dx$ (iv) $\int_{-\pi}^{\pi} \sec^8 x \tan x\, dx$

(v) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x\, dx$ (vi) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\, dx$

(vii) $\int_{-2}^{2} (x^4 + 2x^2)\, dx$

4. Given that $\frac{d}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{(1+x^2)^{\frac{3}{2}}}$ then evaluate $\int_0^3 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$

5. Given that $\frac{d}{dx}[F(x)] = \frac{2+x^2}{1+x^2}$, evaluate $F(\sqrt{3}) - F(1)$. If $F(1) = \pi$, find $F(x)$.

6. Given that $\int_{-2}^{3} f(x)\,dx = 4$ and $\int_{5}^{3} f(x)\,dx = 7$ then evaluate by using suitable properties.

(i) $\int_{3}^{-2} f(x)\,dx$ (ii) $\int_{-2}^{3} f(y)\,dy$ (iii) $\int_{3}^{5} f(y)\,dy$

(iv) $\int_{2}^{2} f(x)\,dx$ (v) $\int_{-2}^{5} f(x)\,dx$ (vi) $\int_{5}^{-2} f(y)\,dy$

7. Evaluate the following integrals by using trigonometric substitutions.

(i) $\int_{0}^{2} \frac{dx}{\sqrt{16-x^2}}$ (ii) $\int_{-1}^{1} \frac{dx}{4-x^2}$

(iii) $\int_{3}^{2\sqrt{3}} \frac{x^3 dx}{\sqrt{x^2+4}}$ (iv) $\int_{0}^{\sqrt{3}} x^2\sqrt{3-x^2}\,dx$

8. Compute the definite integrals by using integration by parts.

(i) $\int_{5}^{9} xe^{4x}\,dx$ (ii) $\int_{1}^{4} x^2 \ln x\,dx$

(iii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin 2x\,dx$ (iv) $\int_{0}^{1} \tan^{-1} x\,dx$

6.6.4 Represent definite integral as the area under the curve

Let $y = f(x)$ is the equation of the curve as shown in the figure 6.5. Suppose $x = a$ and $x = b$ be two vertical lines on x-axis. To determine the area under the curve and above the x-axis between $x = a$ and $x = b$, we divide the region into n small rectangular strips each of with Δx.

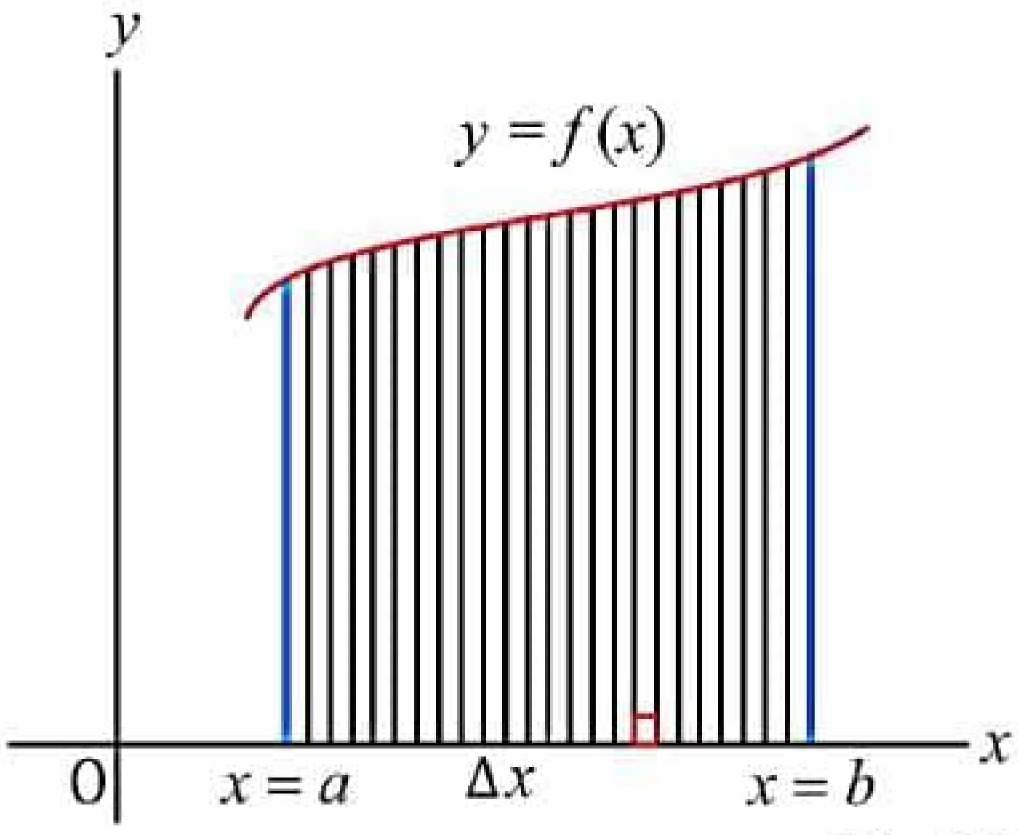


Fig. 6.5

The area of a small rectangular strip of width Δx

$$\Delta A = f(x)\Delta x$$

The total area A bounded by the curve, above x-axis will be equal to the sum of the areas of each rectangular strip from $x = a$ to $x = b$.

$$A = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

If $\Delta x \to 0$ and $n \to \infty$ by using summation notation.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \text{ where } i = 1, 2, 3, \ldots, n$$

By definition of definite integral it can be written as

$$A = \int_a^b f(x)\,dx$$

or $$\boxed{A = \int_a^b y\,dx}$$

Thus, the definite integral represents the area under the curve.

Notes: If $y = f(x) \geq 0$, then A is above the x-axis.
If $y = f(x) \leq 0$, then A is below the x-axis.

6.6.5 Apply definite integrals to calculate area under the curve

Example 1. Find the area, above the x-axis under the following curves between the given ordinates:

(i) $y = x^2 + 1\ ,\ x = 2\ ,\ x = 4$

Solution: By using formula

$$A = \int_a^b y\,dx$$

$$A = \int_2^4 (x^2+1)\,dx$$

$$A = \left[\frac{x^3}{3} + x\right]_2^4$$

$$A = \frac{1}{3}(64-8) + (4-2)$$

$$A = \frac{56}{3} + 2$$

$$A = \frac{62}{3}$$

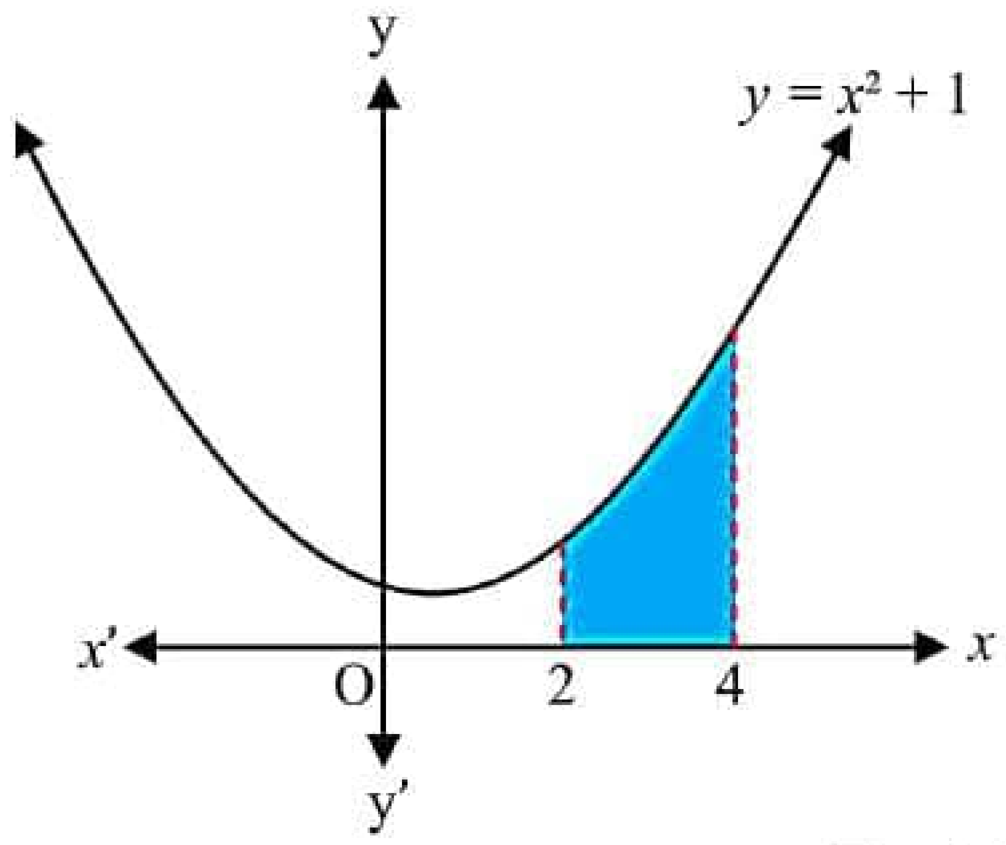


Fig. 6.6

(ii) $y = \cos 3x\ ,\ x = 0\ ,\ x = \frac{\pi}{6}$

Solution: By using formula

$$A = \int_a^b y\,dx$$

$$A = \int_0^{\frac{\pi}{6}} \cos 3x\,dx$$

$$A = \left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}}$$

$$A = \frac{1}{3}\left[\sin 3\left(\frac{\pi}{6}\right) - \sin 0\right]$$

$$A = \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$A = \frac{1}{3}(1 - 0)$$

$$A = \frac{1}{3}$$

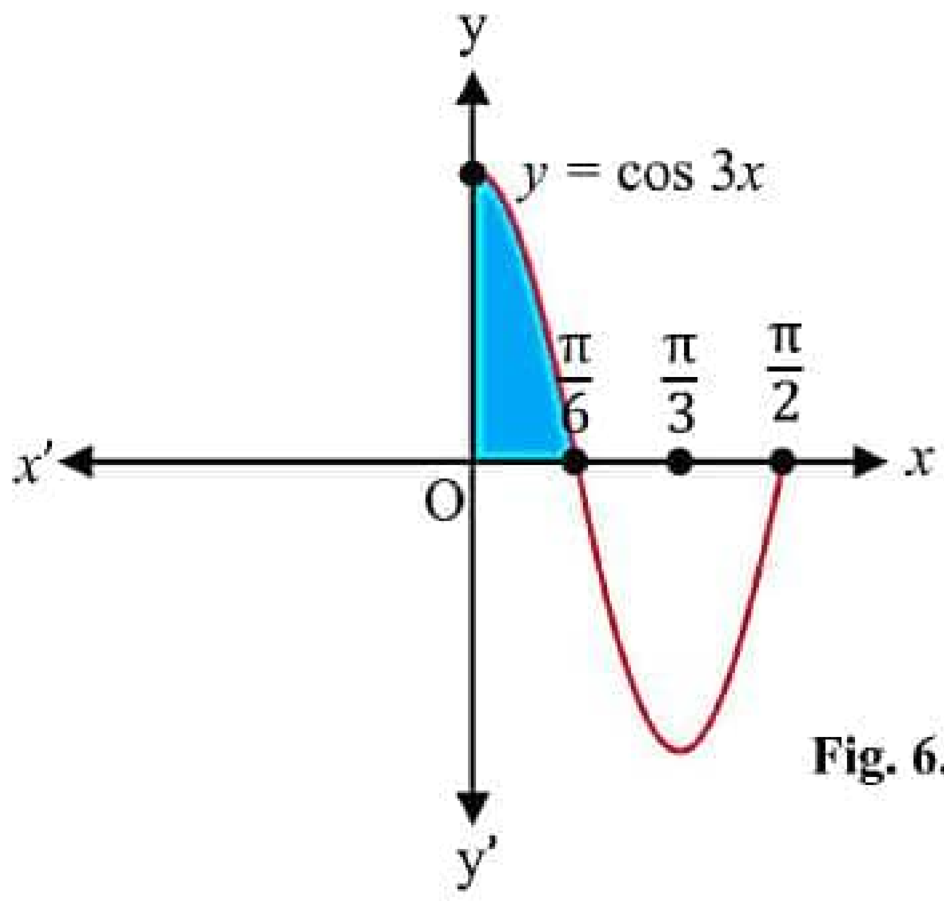


Fig. 6.7`

(iii) $x^2 + y^2 = 16$, $x = 1$, $x = 3$

Solution:

$$x^2 + y^2 = 16$$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

We need area above the x-axis, so we take positive branch of the relation $y = \pm\sqrt{16 - x^2}$

i.e., $y = \sqrt{16 - x^2}$

By using formula

$$A = \int_a^b y\,dx$$

$$A = \int_1^3 \sqrt{16 - x^2}\,dx$$

$$A = \frac{1}{2}\left[x\sqrt{16 - x^2}\right]_1^3 + \left[\frac{1}{2}(16)\sin^{-1}\left(\frac{x}{4}\right)\right]_1^3$$

$$A = \frac{1}{2}\left[3\sqrt{16 - 9} - \sqrt{16 - 1}\right] + 8\left[\sin^{-1}\left(\frac{3}{4}\right) - \sin^{-1}\left(\frac{1}{4}\right)\right]$$

$$A = \frac{1}{2}\left[3\sqrt{7} - \sqrt{15}\right] + 8(0.848 - 0.252)$$

$$A = 2.03 + 4.76$$

$$A = 6.8 \text{ approx.}$$

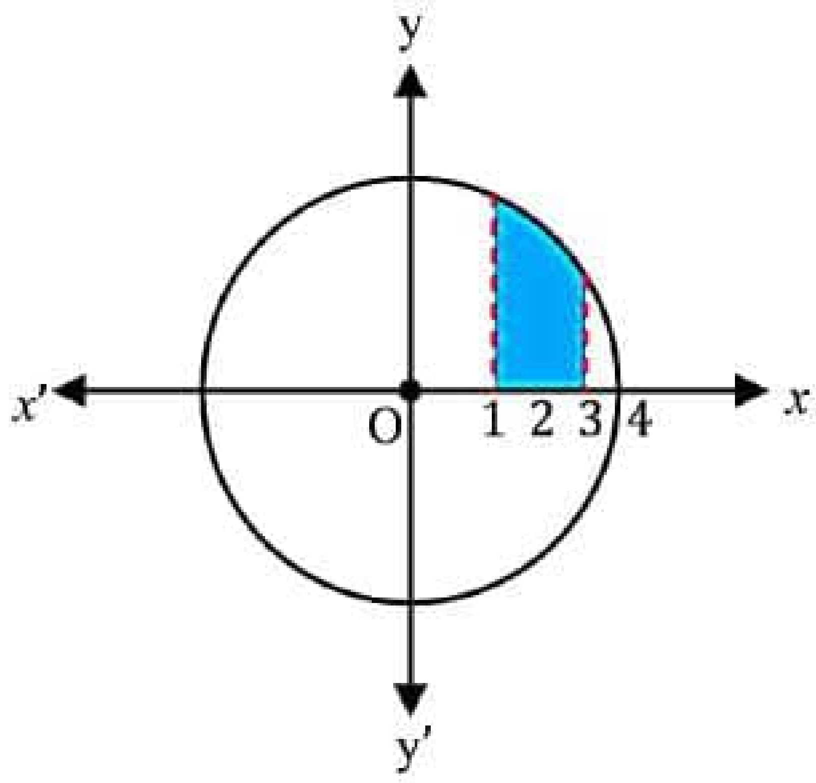


Fig. 6.8

6.6.6 Use MAPLE command int to evaluate definite and indefinite integrals.

The format of int command in MAPLE is as under:

```
> Int(f,x=a..b)
```

where,

f is the function whose definite integral is required x = a..b is the definite integral with lower limit 'a' and upper limit 'b'

In order to compute the integral of a function under definite interval, following examples are given:

$> int(x^3 + 1, x = 1..2)$

$$\frac{19}{4}$$

$> int(x^3 + 1, x = 1..3)$

$$22$$

$> int(x^3 + x^2 + 3x + 4x, x = 1..2)$

$$\frac{175}{12}$$

$> int(e^{3x+1}, x)$

$$\frac{1}{3}e^{3x+1}$$

$> int(e^{3x+1}, x = 0..2)$

$$-\frac{1}{3}e + \frac{1}{3}e^7$$

$> int\left(\sin(x), x = 0..\frac{\pi}{2}\right)$

$$1$$

$> int(\sin(x), x = 0..\pi)$

$$2$$

$> int\left(\cos(x), x = -\frac{\pi}{2}..\frac{\pi}{2}\right)$

$$2$$

$> int(\ln(x+1), x = 0..1)$

$$-1 + 2\ln(2)$$

$> int\left(\frac{1}{\sqrt{1-x^2}}, x\right)$

$$\arcsin(x)$$

$> int(e^{\cos x}\sin x, x)$

$$\frac{(\cos x - 1)e^{\cos x}\sin}{\cos^2}$$

Exercise 6.7

Find the area, above the x-axis under the following curves, between the given ordinates.

1.	$y = 3x^2 + 2$	$x = 1\ ,\ x = 2$
2.	$y = \frac{1}{\sqrt{4-x^2}}$	$x = \frac{1}{2}\ ,\ x = \frac{\sqrt{3}}{2}$
3.	$y = \ln x$	$x = 1\ ,\ x = 3$
4.	$y = x\sin x$	$x = \frac{\pi}{3}\ ,\ x = \frac{\pi}{2}$
5.	$y = \frac{1}{9+x^2}$	$x = -\sqrt{3}\ ,\ x = \sqrt{3}$
6.	$y = 4x^3 + 3x^2 + 2x + 1$	$x = 0\ ,\ x = 2$
7.	$y = 3\sec^2 x$	$x = \frac{\pi}{6}\ ,\ x = \frac{\pi}{3}$
8.	$y = 6\sin^2 x$	$x = 0\ ,\ x = \frac{\pi}{3}$
9.	$y = 5e^{5x}$	$x = -2\ ,x = 3$
10.	$y = \cos^4 x$	$x = 0\ ,\ x = \frac{\pi}{2}$
11.	$y = \frac{4}{\sin^2 x}$	$x = \frac{\pi}{6},\ x = \frac{\pi}{4}$

12. $x^2 + y^2 = 36$ $\qquad$ $x = -1\ ,\ x = 1$

13. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ $\qquad$ $x = -1\ ,\ x = 1$

14. $y^2 = 2x + 5$ $\qquad$ $x = 1\ ,\ x = 2$

15. $y^2 = \tan^4 x$ $\qquad$ $x = \dfrac{\pi}{6},\ x = \dfrac{\pi}{4}$

16. Write MAPLE Command to find integration of the following functions:

(i) $f(x) = e^{2x}$ $\qquad$ (ii) $f(x) = \sin x$

(iii) $f(x) = \cos 2x$ (iv) $\qquad$ (iv) $f(x) = \ln(1 + x)$ $\qquad$ (v) $f(x) = \dfrac{1}{x}$

Review Exercise 6

1. Choose the correct option.

(i) $\int f^n(x) f'(x) dx$, where $n \neq -1$, is

(a) $\dfrac{f^{n+1}(x)}{n+1}$ $\qquad$ (b) $\dfrac{f^{n-1}(x)}{n-1} + c$

(c) $nf^{n-1}(x) + c$ $\qquad$ (d) $\dfrac{f^{n+1}(x)}{n+1} + c$

(ii) $\int f^n(x) f'(x) dx$, where $n = -1$, is

(a) $\dfrac{f^{n+1}(x)}{n+1} + c$ $\qquad$ (b) $\ln|f^n(x)| + c$

(c) $\ln|f(x)| + c$ $\qquad$ (d) $nf^{n-1}(x) + c$

(iii) $\int x^n dx$, where $n = -1$ is

(a) $\dfrac{x^{n+1}}{n+1} + c$ $\qquad$ (b) $nx^{n-1} + c$

(c) $\dfrac{x^{n-1}}{n-1} - c$ $\qquad$ (d) $\ln x + c$

(iv) $\int \sin x \cos x\, dx =$

(a) $\sin x + c$ $\qquad$ (b) $\cos x + c$

(c) $\dfrac{1}{4}\cos 2x + c$ $\qquad$ (d) $-\dfrac{1}{4}\cos 2x + c$

(v) $\int x^2 \ln e^{x^2} dx =$

(a) $\dfrac{x^2}{4} + c$ $\qquad$ (b) $\dfrac{x^5}{5} + c$

(c) $\ln e^{x^2} + c$ $\qquad$ (d) $\ln x^x + c$

(vi) $\int e^{\ln x^3}\,dx =$

(a) $e^{x^3}+c$ (b) $\frac{x^3}{3}+c$

(c) $\frac{x^4}{4}+c$ (d) $\ln x^3+c$

(vii) $\int (1+\tan^2 x)\,dx =$

(a) $\tan x + c$ (b) $\sin^2 x + c$

(c) $\frac{\tan^2 x}{2}+c$ (d) $\ln \sec x + c$

8. $\int \frac{2e^x}{1+e^x}\,dx =$

(a) $\ln(1+e^x)+c$ (b) $\ln(1+e^x)^2+c$

(c) $(1+e^x)^{-2}+c$ (d) $\frac{e^x}{2}+c$

9. $\int \frac{e^{x+3\ln x}}{x^3}\,dx =$

(a) $\frac{1}{3}e^{x+3\ln x}+c$ (b) e^x+c

(c) $e^{x+3\ln x}+c$ (d) $3\ln x + c$

10. $\int \ln\left(e^x \cdot e^{\sin x}\right)dx =$

(a) $\frac{1}{e^{x+\sin x}}+c$ (b) $\ln \sin x + c$

(c) $\frac{x^2}{2} - \cos x + c$ (d) $x \ln \sin x + c$

11. $\int \frac{\frac{1}{x\sqrt{x^2-1}}}{cosec^{-1}x}\,dx =$

(a) $\ln(\text{cosec}^{-1}x)^{-1}+c$ (b) $(\text{cosec}^{-1}x)^2+c$

(c) $\text{cosec}^{-1}x+c$ (d) $\ln(\text{cosec}^{-1}x)+c$

12. If $F(x)$ is an antiderivative of $f(x)$ then

$\int_a^b f(x)\,dx =$

(a) $F(a)-F(b)$ (b) $F(b)-F(a)$

(c) $f(b)-f(a)$ (d) $\frac{f(b)}{f(a)}$

13.

$$\int_{-50}^{+50} (x^3 + x)\, dx$$

(a) 0 (b) 1000 (c) 2000 (d) 3000

14.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^9 x \cos^{11} x\, dx$$

(a) 1 (b) 3 (c) 0 (d) $2\int_0^{\frac{\pi}{2}} \sin^9 x \cos^{11} x\, dx$

15.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{10} x \cos^{11} x\, dx$$

(a) 0 (b) 1 (c) 3 (d) $2\int_0^{\frac{\pi}{2}} \sin^{10} x \cos^{11} x\, dx$

16. Area bounded by the curve $y = \ln e^{x^2}$ from $x = -1$ to $x = 1$ is

(a) $\frac{2}{3}$ (b) 1 (c) $\ln 2$ (d) $\ln 3$

17. $\int_a^b f(x)\, dx =$

(a) $-\int_a^b f(x)\, dx$ (b) $\int_b^a f(x)\, dx$

(c) $-\int_b^a f(x)\, dx$ (d) 0

18. $\int_2^2 \left(x^3 + 3x^2 - 5x^{-\frac{1}{2}}\right) dx =$

(a) 0 (b) $12 - 10\sqrt{2}$

(c) $24 - 20\sqrt{2}$ (d) $20\sqrt{2}$

19. $\int_{-2}^{2} (x^5 - x^3 + x)^5\, (5x^4 - 3x^2 + 1) dx =$

(a) $\frac{1}{3}(26)^6$ (b) 0

(c) $2(26)^6$ (d) $\frac{(26)^6}{6}$

20. $\int \frac{dx}{a^2+x^2} =$

(a) $tan^{-1}\frac{x}{a}$ (b) $\frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(c) $\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (d) $\sin^{-1}\frac{x}{a}+c$

21. Evaluate the following integrals.

(i) $\int 3x^5 dx$

(ii) $\int x \ln x^n \, dx$

(iii) $\int \sec^5 x \, dx$

(iv) $\int \frac{y^2}{\sqrt{1-y^2}} dy$

(v) $\int \sqrt{1-\sin 2x} \, dx$

(vi) $\int \tan^5 x \sec^{\frac{5}{2}} x \, dx$

(vii) $\int \frac{\cos x dx}{(2+\sin x)(3+\sin x)}$

(viii) $\int x^2 \sin x \, dx$

(ix) $\int \frac{d\theta}{\sqrt{1+\cos\frac{5\theta}{2}}}$

(x) $\int \frac{dx}{x^2-81}$

(xi) $\int \cot^5 x \operatorname{cosec}^{\frac{3}{2}} x \, dx$

(xii) $\int \cos^5 x \sin^5 x \, dx$

(xiii) $\int_0^2 6(x^2+3x+2)^5 (2x+3) dx$

(xiv) $\int_0^{\frac{\pi}{2}} \cos^3 x \sqrt{\sin x} \, dx$

(xv) $\int_{-a}^{a} \frac{dx}{x\sqrt{x^2-a^2}}$

(xvi) $\int_0^a \frac{dx}{x^2+a^2}$

Plane Analytic Geometry: Straight Line

Unit 7

Analytic geometry utilizes the concepts of algebra to locate the position of a point on the plane using an ordered pair of numbers. It can be understood as a combination of geometry and algebra. In analytic geometry different algebraic equations are used to describe the dimension and position of different geometric figures.

7.1 Division of a Line Segment

7.1.1 Recall distance formula to calculate distance between two points given in Cartesian plane

We know that, the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found by

$$d = |\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

known as distance formula (Fig. 7.1).

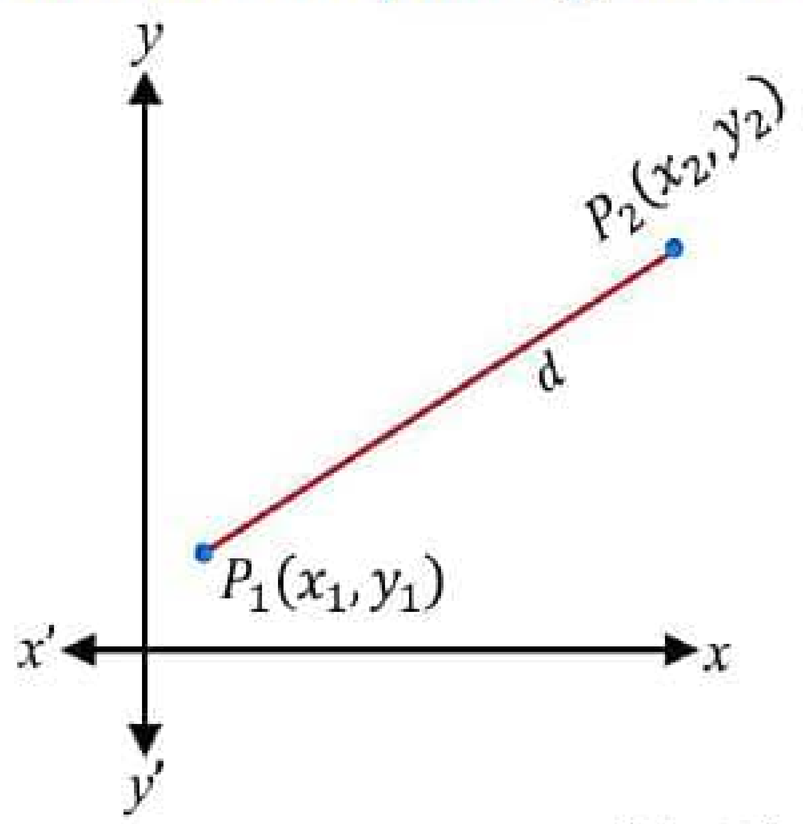


Fig 7.1

Example: Find the distance between two points $P(1,7)$ and $Q(-2,\ 3)$.

Solution: The distance between the given points P and Q having coordinates $(1,7)$ and $(-2,3)$, by using the distance formula;

$$\begin{aligned} d = |\overline{PQ}| &= \sqrt{(-2-1)^2 + (3-7)^2} \\ &= \sqrt{(-3)^2 + (-4)^2} \\ &= \sqrt{9 + 16} \\ &= 5 \text{ units} \end{aligned}$$

7.1.2 Recall Mid-point formula

In previous class, we have learnt the mid-point formula to find the mid-point of a line segment when its end points are given. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are the end points of a line segment then the mid-point $C(x, y)$ of $\overline{AB}$ is found by

$$C(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

As shown in the figure 7.2.

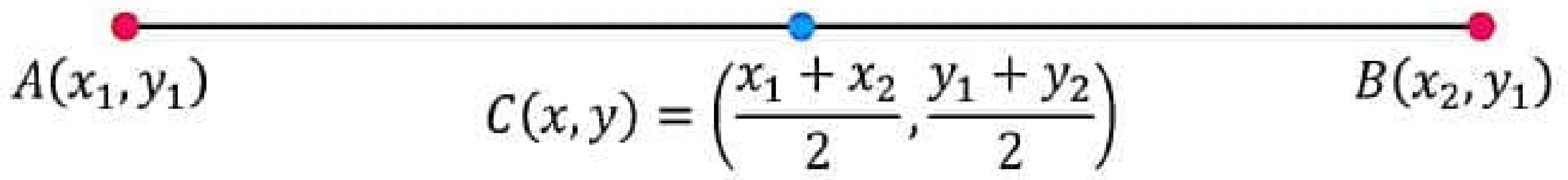


Fig. 7.2

Mid-point $C(x, y)$ divides the $\overline{AB}$ into two equal parts.

For example, the mid-point of line segment $\overline{AB}$ whose endpoints are $(-1, 3)$ and $(5, -3)$ will be

$$\left(\frac{-1+5}{2}, \frac{3+(-3)}{2}\right) = (2, 0)$$

7.1.3 Find coordinates of a point that divides the line segment in given ratio (internally and externally)

Let $\overline{AB}$ is the line segment and A, B and C are the collinear points then C will divide the line segment AB internally or externally in the ratio of m and n.

Let us find the coordinates of division point for both cases.

Internal division

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in the xy-plane. Let $C(x, y)$ be the point which divides line segment AB internally in the ratio $m{:}n$. In Fig. 7.3 $\overline{AP}$, $\overline{CN}$ and $\overline{BR}$ are drawn perpendiculars to x-axis. AS and $\overline{CT}$ are drawn parallel to x-axis.

$m\angle CAS = m\angle BCT$ (corresponding angles)

$m\angle CSA = m\angle BTC = 90°$

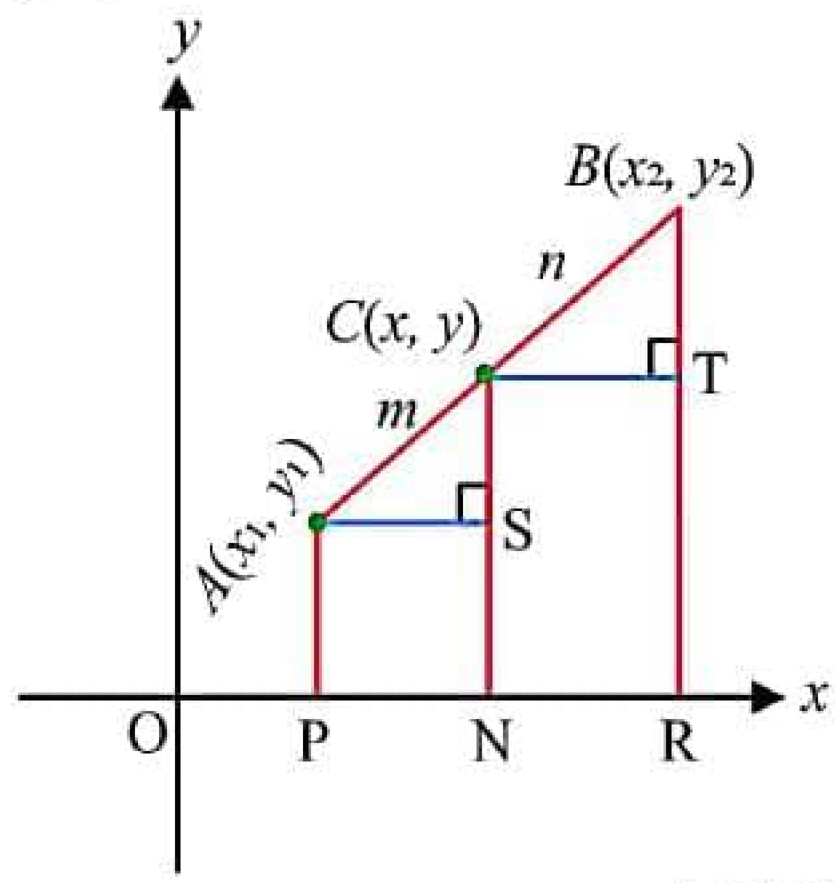
By similarity criterion of triangle

Similarly,

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $\frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$$\Rightarrow y = \frac{my_2 + ny_1}{m + n}$$



$\Delta CAS \sim \Delta BCT$

Fig 7.3

$$\Rightarrow \frac{|\overline{AC}|}{|\overline{CB}|} = \frac{|\overline{AS}|}{|\overline{CT}|} = \frac{|\overline{CS}|}{|\overline{BT}|} = \frac{m}{n} \quad \text{... (i)}$$

Now,

$$|\overline{AS}| = |\overline{PN}| = |\overline{ON}| - |\overline{OP}| = x - x_1$$

$$|\overline{CT}| = |\overline{NR}| = |\overline{OR}| - |\overline{ON}| = x_2 - x$$

$$|\overline{CS}| = |\overline{CN}| - |\overline{SN}| = y - y_1$$

$|\overline{BT}| = |\overline{BR}| - |\overline{RT}| = y_2 - y$

From equation (i),

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $\frac{m}{n} = \frac{x-x_1}{x_2-x}$

$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m + n}$

Similarly,

$$\frac{m}{n} = \frac{y - y_1}{y_2 - y} = \frac{x - x_1}{x_2 - x}$$

Take, $\frac{m}{n} = \frac{y-y_1}{y_2-y}$

$\Rightarrow \quad y = \frac{my_2 + ny_1}{m + n}$

So, the coordinates of the point $C(x, y)$ which divides the line segment joining points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$ are $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ that is known as section formula.

External Division:

Let $D(x, y)$ divides $\overline{AB}$ externally in the ratio $m:n$, where $m:n = \frac{|\overline{AD}|}{|\overline{BD}|}$. Now draw perpendiculars from $A(x_1, y_1)$, $B(x_2, y_2)$ and $D(x, y)$, along coordinate axes, which meet at $S(x, y_2)$ and $R(x, y_1)$. Therefore, there exist two similar right triangles ADR and BDS, then by similar triangles as shown in Fig. 7.4.

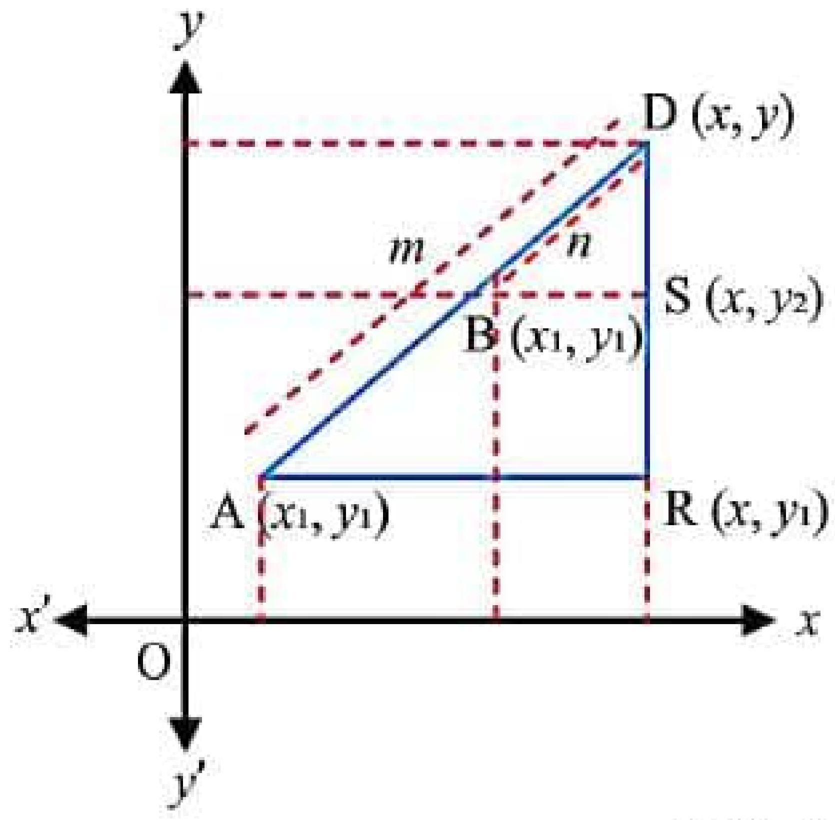


Fig 7.4

Now,

$$\frac{m}{n} = \frac{|\overline{AD}|}{|\overline{BD}|} = \frac{|\overline{AR}|}{|\overline{BS}|} = \frac{x - x_1}{x - x_2}$$

$\therefore \quad \frac{m}{n} = \frac{x - x_1}{x - x_2}$

$\Rightarrow \quad x = \frac{mx_2 - nx_1}{m - n}$

Similarly,

$$\frac{m}{n} = \frac{|\overline{AD}|}{|\overline{BD}|} = \frac{|\overline{DR}|}{|\overline{DS}|} = \frac{y - y_1}{y - y_2}$$

$\therefore \quad \frac{m}{n} = \frac{y - y_1}{y - y_2}$

$\Rightarrow \quad y = \frac{my_2 - ny_1}{m - n}$

Hence, coordinate of D which divides externally in the ratio $m_1 : m_2$ are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$$

Where $m - n \neq 0$.

Note: When point P divides the line segment AB internally, the given ratio $m:n$ will be positive and for external division the ratio will be negative.

Example 1. Find the coordinates of the point, which divides the line segment joining the points $(3,2)$ and $(4,-5)$ internally in the ratio $3:2$.

Solution: Here $(x_1, y_1) = (3,2)$ and $(x_2, y_2) = (4,-5)$. Also $m:n = 3:2$.

By using the section formula;

Point of division is

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{3 \times 4 + 2 \times 3}{3+2}, \frac{3 \times -5 + 2 \times 3}{3+2}\right)$$

$$= \left(\frac{18}{5}, \frac{-11}{5}\right)$$

Example 2. Find the point of division of the line segment joining $(1,-2)$ to $(-3,4)$ externally in the ratio $3:5$.

Solution: Here $(x_1, y_1) = (1,-2)$ and $(x_2, y_2) = (-3,4)$. Also $m:n = 3:2$

$$P(x,y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

$$= \left(\frac{(3)(-3) - 5(-2)}{3-5}, \frac{3(4) - 5(-2)}{3-5}\right)$$

$$= (7,-11)$$

Example 3. If $A(2,4), B(4,5), C(p,q)$ and $D((1,3)$ are the vertices of parallelogram then find the values of p and q.

Solution: We know that the diagonals of a parallelogram bisect each other. Let O be the point at which diagonals intersect. Coordinates of the midpoints (x,y) of both line segments AC and BD will be same. Thus, using midpoint formula;

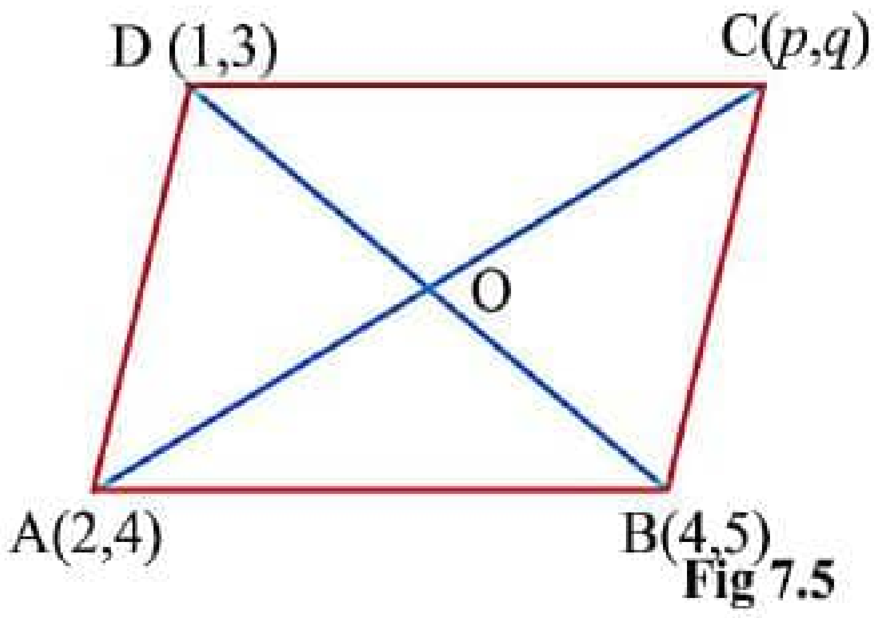


Fig 7.5

Point of division

Mid-point of $\overline{BD}$ = mid-point of $\overline{AC}$

$$\left(\frac{4+1}{2}, \frac{5+3}{2}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{5}{2}, \frac{8}{2}\right) = \left(\frac{2+p}{2}, \frac{4+q}{2}\right)$$

$$\left(\frac{5}{2}, 4\right) = \left(\frac{2+p}{2}, \frac{4+q}{2}\right)$$

Thus,

$$\frac{5}{2} = \frac{2+p}{2}$$

$$\Rightarrow p = 3$$

and

$$4 = \frac{4+q}{2}$$

$\Rightarrow q = 4$

Let us recall the definitions of some important terms which will help us to show many useful results.

Point of Concurrency: A point where three or more lines or rays intersect with each other is known as the point of concurrency.

Perpendicular bisector: A line segment which bisects another line segment at 90° is called perpendicular bisector.

Angle Bisector: An angle bisector is a straight-line drawn from the vertex of a triangle to its opposite side in such a way, that it divides the angle into two equal or congruent angles.

Median: Line segment joining a vertex to the mid-point of the side opposite to that vertex is called the median of a triangle.

Altitude: The altitude of a triangle is the perpendicular line segment drawn from the vertex to the opposite side of the triangle.

As four different types of line segments can be drawn to a triangle, therefore there will be four different points of concurrency in a triangle. Such concurrent points are referred to as different centers according to the lines meeting at that point. The four different points of concurrency in a triangle are:

Circumcentre: The point where three perpendicular bisectors of the triangle meet is called circumcentre of the triangle.

Incentre: The point where three angle bisectors of the triangle meet is called incenter of the triangle.

Centroid: The point where three medians of the triangle meet is called centroid of the triangle.

Orthocentre: The point where three altitudes of the triangle meet is called orthocentre of the triangle.

7.1.4 Show that the medians and angle bisectors of a triangle are concurrent

Show that the Medians of a Triangle are concurrent

Proof: Let ABC is a triangle (Fig. 7.6) with medians $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ respectively where F is the midpoint of line segment BC, D of $\overline{AB}$ and E of $\overline{AC}$ respectively.

The midpoint of side $\overline{BC}$ is

$$F = \frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}$$

The midpoint of side $\overline{AB}$ is

$$D = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint of side $\overline{AC}$ is

$$E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$$

In triangle ABC, say P is the point of intersection. The coordinates of point P that divides the $\overline{AF}$ in the ratio 2: 1 are as under:

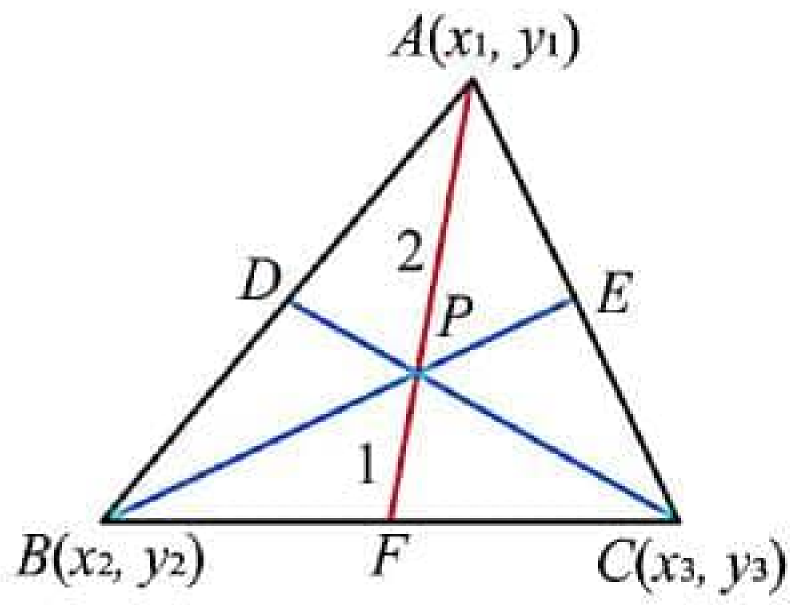


Fig 7.6

Let $(x_1, y_1) = (x_1, y_1)$ and

$(x_2, y_2) = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ because F is midpoint of $\overline{BC}$.

$$P(x,y) = \left(\frac{(1)x_1 + (2)\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{(1)y_1 + (2)\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$$

$$P(x,y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Again, the coordinates of point $P(x, y)$ that divides the $\overline{BE}$ in the ratio 2: 1 are as under:

Let $(x_1, y_1) = (x_2, y_2)$ and $(x_2, y_2) = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ because E is midpoint of $\overline{AC}$.

Thus,

$$P(x,y) = \left(\frac{(1)x_1 + (2)\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{(1)y_1 + (2)\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$$

$$P(x,y) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly, the coordinates of point $P(x, y)$ that divides the $\overline{CD}$ in the ratio 2: 1 are as under:

Let $(x_1, y_1) = (x_3, y_3)$ and $(x_2, y_2) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ because D is midpoint of AB.

Thus,

$$P(x,y) = \left(\frac{(1)x_3 + (2)\left(\frac{x_1+x_2}{2}\right)}{1+2}, \frac{(1)y_3 + (2)\left(\frac{y_1+y_2}{2}\right)}{1+2}\right)$$

$$P = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Hence, the medians of the triangle are concurrent. It is called centroid of the triangle.

Example: Find the point of concurrency of medians of triangle ABC where coordinates of A, B and C are $(4, 10)$, $(8, 2)$ and $(-8, 4)$.

Solution: Here, $(x_1, y_1) = (4, 10)$, $(x_2, y_2) = (8, 2)$ and $(x_3, y_3) = (-8, 4)$

The point of concurrency of the points $(4, 10)$, $(8, 2)$ and $(-8, 4)$ is

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$= \left(\frac{4 + 8 - 8}{3}, \frac{10 + 2 + 4}{3}\right) = \left(\frac{4}{3}, \frac{16}{3}\right)$$

Show that angle bisectors of triangle are concurrent

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of any ΔABC. Let a, b and c be the measures of the sides $\overline{BC}, \overline{AC}$ and $\overline{AB}$ respectively as shown in the figure 7.7.

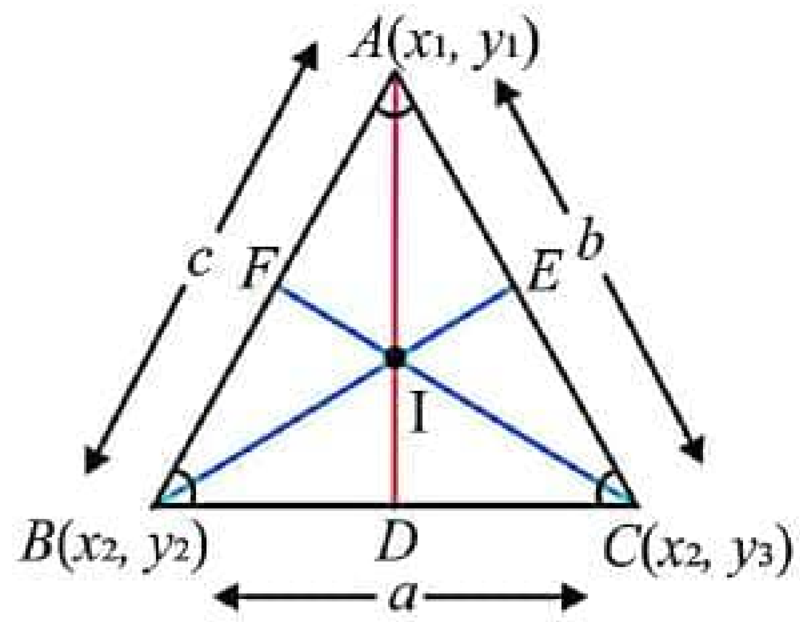


Fig 7.7

Let $\overline{AD}$ be the angle bisector of $\angle A$ which divides $\overline{BC}$ at D internally in the ratio of the sides containing the angle.

i.e., $$\frac{|\overline{BD}|}{|\overline{DC}|} = \frac{c}{b} \quad \ldots\text{(i)}$$

$$\therefore \quad D(x, y) = \left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$$

Also angle bisector of $\angle$B divides $\overline{AD}$ at I in $|\overline{BD}|:|\overline{AB}|$

$$\therefore \quad \frac{|\overline{AB}|}{|\overline{BC}|} = \frac{c}{|\overline{BC}|} \quad \ldots\text{(ii)}$$

From (i), we have

$$\frac{|\overline{BD}|}{|\overline{DC}|} = \frac{c}{b}$$

By componendo property

$$\frac{|\overline{BD}|}{|\overline{BD}| + |\overline{DC}|} = \frac{c}{c + b}$$

$$\frac{|\overline{BD}|}{|\overline{BC}|} = \frac{c}{c + b}$$

$$|\overline{BD}| = \frac{ac}{b + c} \quad [\because |\overline{BC}| = a]$$

From (ii), we have

$$\frac{|\overline{AB}|}{|\overline{BD}|} = \frac{c}{|\overline{BD}|} = \frac{c}{\frac{ac}{b + c}} = \frac{b + c}{a} = (b + c) : a$$

i.e., I divides $|\overline{AD}|$ internally in the ratio $(b+c):a$

$$\therefore \quad I(x,y)=\left(\frac{ax_1+(b+c)\cdot\left(\frac{bx_2+cx_3}{b+c}\right)}{a+(b+c)},\frac{ay_1+(b+c)\cdot\left(\frac{by_2+cy_3}{b+c}\right)}{a+(b+c)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \ldots\text{(i)}$$

Similarly,

$$E(x,y)=\left(\frac{ax_1+cx_3}{a+c},\frac{ay_1+cy_3}{a+c}\right)$$

Point I divides $\overline{BE}$ internally in the ratio $(a+c):b$

$$\therefore \quad I(x,y)=\left(\frac{bx_2+(a+c)\cdot\left(\frac{ax_1+cx_3}{a+c}\right)}{b+(a+c)},\frac{by_2+(a+c)\cdot\left(\frac{ay_1+cy_3}{a+c}\right)}{b+(a+c)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \ldots\text{(ii)}$$

Again,

$$F(x,y)=\left(\frac{ax_1+bx_2}{a+b},\frac{ay_1+by_2}{a+b}\right)$$

Point I divides $\overline{CF}$ internally in the ratio $(a+b):c$

$$\therefore \quad I(x,y)=\left(\frac{cx_3+(a+b)\cdot\left(\frac{ax_1+bx_2}{a+b}\right)}{c+(a+b)},\frac{cy_3+(a+b)\cdot\left(\frac{ay_1+by_2}{a+b}\right)}{c+(a+b)}\right)$$

$$\Rightarrow \quad I(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \ldots\text{(iii)}$$

From equation (i), (ii) and (iii), the coordinates of point I on each angle bisector are found to be same. This means all angle bisectors passes through I. Thus, angle bisectors of a triangle are concurrent.

Exercise 7.1

1. Find the distance between the following pairs of points:
   (i) $A(-1,3)$ and $B(5,-5)$ (ii) $C(-1,0)$ and $D(0,-1)$
   (iii) $E(1,-1)$ and $F(2,7)$ (iv) $G(-1,-4)$ and $H(5,-4)$
2. Find the point on the y-axis which is $5\sqrt{2}$ units away from $(5,2)$.
3. Find the point on the x-axis which is $\sqrt{41}$ units away form $(-7,5)$.
4. If ABC is triangle whose vertices are $A(-3,3), B(2,6)$ and $C(3,0)$. Give the most specific name for ΔABC.

5. If the point $P(2,1)$ lies on the line segment joining the points $A(4,2)$ and $B(8,4)$, then show that $|\overline{AP}| = \frac{1}{2}|\overline{AB}|$.
6. Find the coordinates of the midpoint of following points:
   (i) $M(-4,2)$ and $N(-4,-2)$ (ii) $P(-1,-4)$ and $Q(5,-4)$
   (iii) $S(1,-2)$ and $T(2,-4)$ (iv) $X(6,2)$ and $Y(-2,-7)$
7. Find the point which divides the line segment joining $(4,-1)$ and $(4,3)$ in the ratio $3:1$ internally.
8. The points $P(-2,2), Q(2,-1)$ and $R(-1,4)$ are the mid-points of the sides of the triangle. Find the vertices.
9. $Z(4,5)$ and $X(7,-1)$ are two given points and the point Y divides the line-segment ZX externally in the ratio $4:3$. Find the coordinates of Y.
10. If a point $P(k,7)$ divides the line segment joining $A(8,9)$ and $B(1,2)$ in a ratio $m:n$ then find ratio $m:n$ also find k.
11. $A(2,7)$ and $B(-4,-8)$ are coordinates of the line segment AB. There are two points that trisect the segment AB. Find the points of trisection.
12. The vertices P, Q and R of a triangle are $(2,1)$, $(5,2)$ and $(3,4)$ respectively. Find the coordinates of the circum-centre and also the radius of the circum-circle of the triangle.
13. $\overline{AB}$ is divided into 20 equal parts by $P_1, P_2, P_3, \ldots P_{10}, \ldots P_{19}$. If A and B are (2,3) and (10,11) respectively, find the coordinates of P_{13}.
14. If A, B and C are three collinear point and the coordinates of A and B are $(3,4)$ and $(7,7)$ respectively. Find the coordinates of C if $|\overline{AC}| = 10$ units.
15. Find the coordinates of the incentre of triangle whose angular points are respectively;
   (i) $L(2,8), M(8,2)$ and $N(9,9)$ (ii) $P(-36,7), Q(20,7)$ and $R(0,-8)$
16. The line segment joining $P(-8,10)$ and $Q(6,-4)$ is cut by x and y axes at A and B respectively. Find the ratios in which A and B divide $\overline{PQ}$.
17. Find the coordinates of the centroid of a triangle whose angular points are
   (i) $A(1,3), (2,7)$ and $5,6)$
   (ii) $P(-2,5) Q\ (-7,1)$ and $R(-8,-4)$.
18. A straight line passes through the points $(7,9)$ and $(-1,1)$. Find a point in the line whose ordinate is 4.

7.2 Slope (Gradient) of a Straight line

Slope or gradient of a line is a number that describes both the direction and the steepness of the line. The concept of slope has many applications in the real world.

In construction, the pitch of a roof, the slant of the plumbing pipes and the steepness of the stairs are few applications of slope.

7.2.1 Define the slope of a line

To understand the definition of the slope, first we understand the inclination of a line.

Inclination of a line:

Inclination of a line is the smallest positive angle between the line and the positive direction of x-axis. In Fig. 7.8, θ is the inclination of l, where θ is $0 < \theta < \pi$.

Note: The inclination of x-axis is taken as 0.

Slope of a line:

Slope of a line is the tangent of its inclination. It is denoted by m.

i.e., $m = \tan\theta$

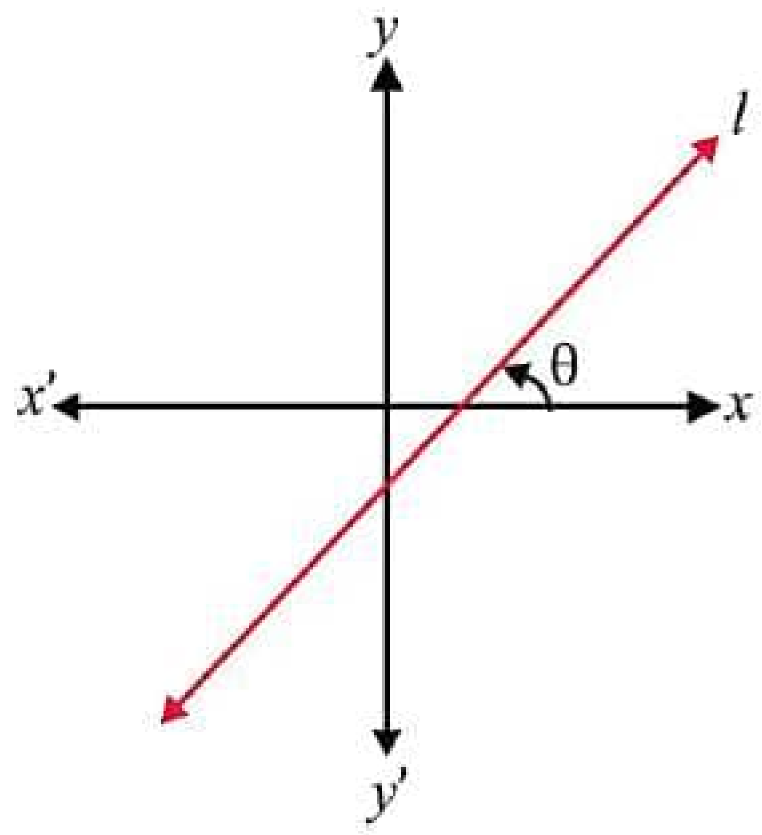


Fig 7.8

7.2.2 Derive the formula to find the slope of a line passing through two points

Let l is a line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in the figure 7.9.

Here θ is the inclination of the line and m is the slope of the line,

i.e., $m = \tan\theta$ ...(i)

The changes in abscissa and ordinates are $x_2 - x_1$ and $y_2 - y_1$ respectively.

Consider the right triangle ABC

$$\tan\theta = \frac{|\overline{BC}|}{|\overline{AC}|} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{...(ii)}$$

By comparing equations (i) and (ii)

We get,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

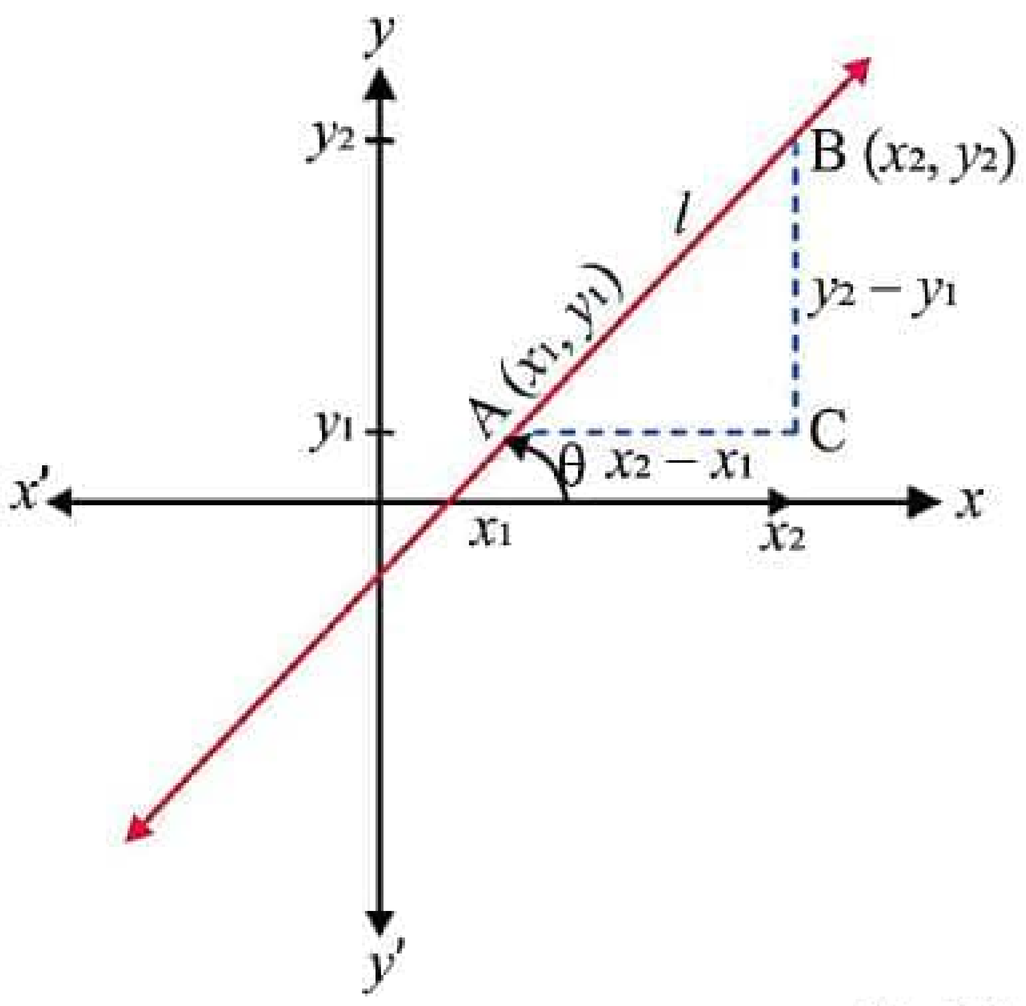


Fig 7.9

The formula states that the slope of a line is equal to the rise over run.

Example 1. Find the slope of a line whose coordinates are $(1, 5)$ and $(4, 7)$.

Solution: Here $(x_1, y_1) = (3, 7)$ and $(x_2, y_2) = (5, 8)$.

We get using Slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow m = \frac{8 - 7}{5 - 3} = \frac{1}{2}$$

Example 2. Find the value of b, if the slope of a line passing through the points $(-2, b)$ and $(3, 4)$ is 5.

Solution: Here, $(x_1, y_1) = (-2, b)$ and $(x_2, y_2) = (3, 4)$.

Using Slope formula = $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$5 = \frac{4 - b}{3 - (-2)}$$

$$5 = \frac{4 - b}{5}$$

$$\Rightarrow \quad 25 = 4 - b$$

$$\Rightarrow \quad b = -21$$

Thus, the value of b is -21.

7.2.3 Find the condition that two straight lines with given slopes may be:

- **parallel to each other,**
- **perpendicular to each other.**

Find the condition that two straight lines with given slopes may be:

- **Parallel to each other**

Let l_1 and l_2 are two straight lines with slopes m_1 and m_2 respectively as shown in the figure 7.10.

Since both lines are parallel to each other, therefore they have same inclination as θ is in our case.

Now, slope of first line

$$m_1 = \tan\theta \qquad \ldots(i)$$

Slope of second line

$$m_2 = \tan\theta \qquad \ldots(ii)$$

From (i) and (ii)

$$m_1 = m_2$$

$\Rightarrow l_1 || l_2$ iff $m_1 = m_2$

Hence two straight lines are parallel to each other iff they have same slopes.

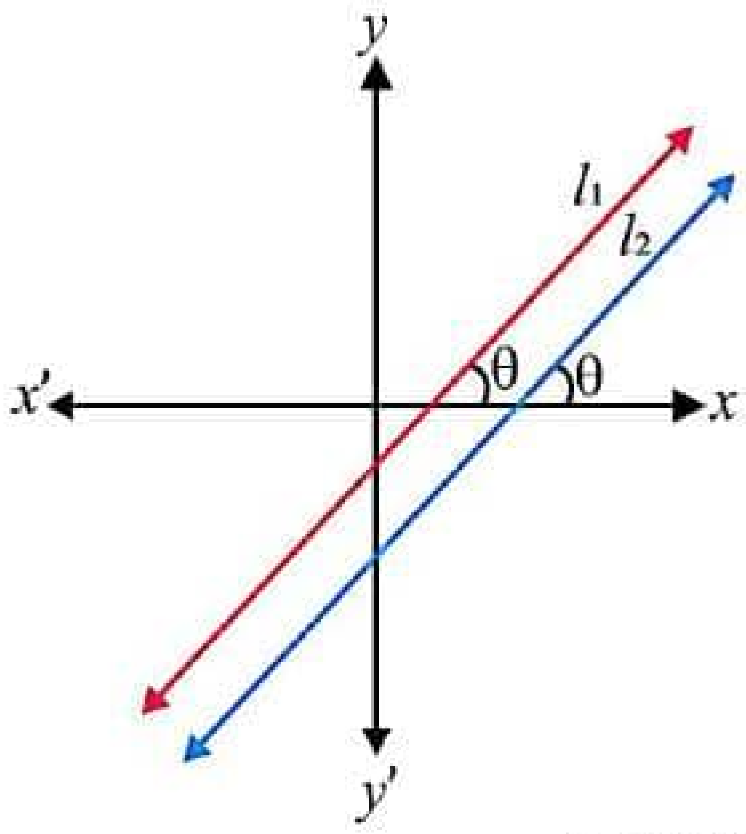


Fig 7.10

- **Perpendicular to each other**

Let l_1 and l_2 are two straight lines perpendicular to each other with slopes m_1 and m_2 respectively. If θ is the inclination of l_1 then $90° + \theta$ will be the inclination of l_2 as shown in the figure 7.11.

Now,

$$m_1 = \tan\theta \qquad \ldots(i)$$

$$m_2 = \tan(90° + \theta)$$

$m_2 = -\cot\theta$

or $m_2 = \dfrac{-1}{\tan\theta}$

$\Rightarrow \tan\theta = \dfrac{-1}{m_2}$ ...(ii)

From equation (ii) and (iii),

we get

$$m_1 = \frac{-1}{m_2}$$

$$\Rightarrow m_1 m_2 = -1$$

$$\Rightarrow l_1 \perp l_2 \text{ iff } m_1 m_2 = -1$$

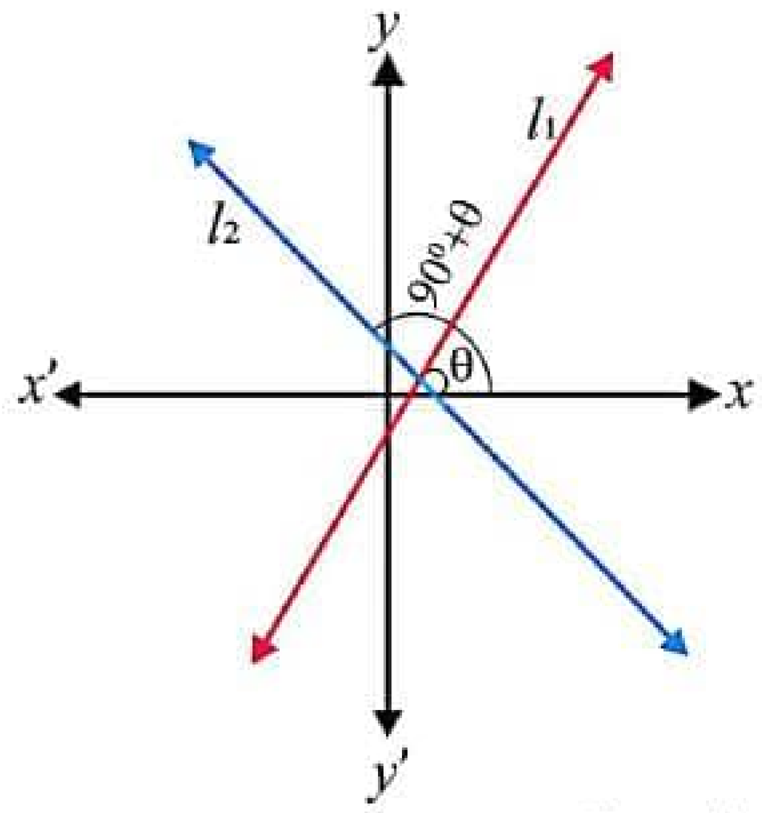


Fig 7.11

Hence two non-vertical straight lines are perpendicular to each other iff product of their slopes is -1.

Example 1. If a line l_1 passes through two given points $(1,3)$ and $(3,7)$ and l_2 passes through $(2,9)$ and $(3,11)$. Check whether both lines are parallel or not.

Solution: Line l_1 passes through two given points $(1,3)$ and $(3,7)$.

The slope of $l_1 = m_1 = \frac{7-3}{3-1} = \frac{4}{2} = 2$

Line l_2 passes through two points $(2,9)$ and $(3,11)$.

The slope of $l_2 = m_2 = \frac{11-9}{3-2} = \frac{2}{1} = 2$

Since, $m_1 = m_2$

Therefore, both lines are parallel to each other.

Example 2. If a line l_1 passes through two given points $(0,1)$ and $(1,-1)$, a line l_2 passes through $(2,2)$ and $(4,3)$. Check whether the following lines are perpendicular or not.

Solution:

The slope of line $l_1 = m_1 = \frac{-1-1}{1-0} = -2$

The slope of line $l_2 = m_2 = \frac{3-2}{4-2} = \frac{1}{2}$

Since $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$

Therefore, both lines are perpendicular to each other.

Exercise 7.2

1. Find the slope of the line passing through given pair of points.

(i)	$A(3,7)$ and $B(2,9)$	(ii)	$C(5,-2)$ and $D(3,6)$
(iii)	$E(5,3)$ and $F(-2,3)$	(iv)	$G(0,0)$ and $H(a,b)$

2. Find the slope of the perpendicular line when the given line passes through the following pair of points.
   (i) $A(2,1)$ and $B(4,5)$ (ii) $C(-1,0)$ and $D(3,5)$
   (iii) $E(2,1)$ and $F(-3,1)$ (iv) $G(-1,2)$ and $H(-1,-5)$
3. In each of the following the slope of the line is given. What is the slope of a line (a) parallel (b) perpendicular, to it?
   (i) $\frac{2}{3}$ (ii) $\frac{-7}{2}$ (iii) -1 (iv) 4
4. Are the lines l_1 and l_2 passing through the given pairs of points parallel, perpendicular or neither?
   (i) $l_1: (1,2), (3,1)$ and $l_2: (0,-1), (2,0)$
   (ii) $l_1: (0,3), (3,1)$ and $l_2: (-1,4), (-7,-5)$
   (iii) $l_1: (2,-1), (5,-7)$ and $l_2: (0,0), (-1,2)$
   (iv) $l_1: (1,0), (2,0)$ and $l_2: (5,-5), (-10,-5)$
5. The line through $(6,-4)$ and $(-3,2)$ is parallel to the line through $(2,1)$ and $(0,y)$. Find y.
6. The line through $(2,5)$ and $(-3,-2)$ is perpendicular to the line through $(4,-1)$ and $(x,3)$. Find x.
7. Using slopes prove that $(-1,4), (-3,-6)$ and $(3,-2)$ are the vertices of right triangle.
8. Using slopes, find the fourth vertex of a rectangle if $(0,-1), (4,-3)$ and $(12,3)$ are its three consecutive vertices.

7.3 Equation of a Straight Line Parallel to Co-ordinate Axes

7.3.1 Find the equation of a straight line parallel to

- **y-axis at a distance a from it,**
- ***x*-axis at a distance b from it.**

- **Line parallel to y-axis at a distance a from it,**

Let l be a line parallel to y-axis at a distance 'a' from it and cutting the axis of x at A, such that $|\overline{OA}| = a$, as shown in the figure 7.12.

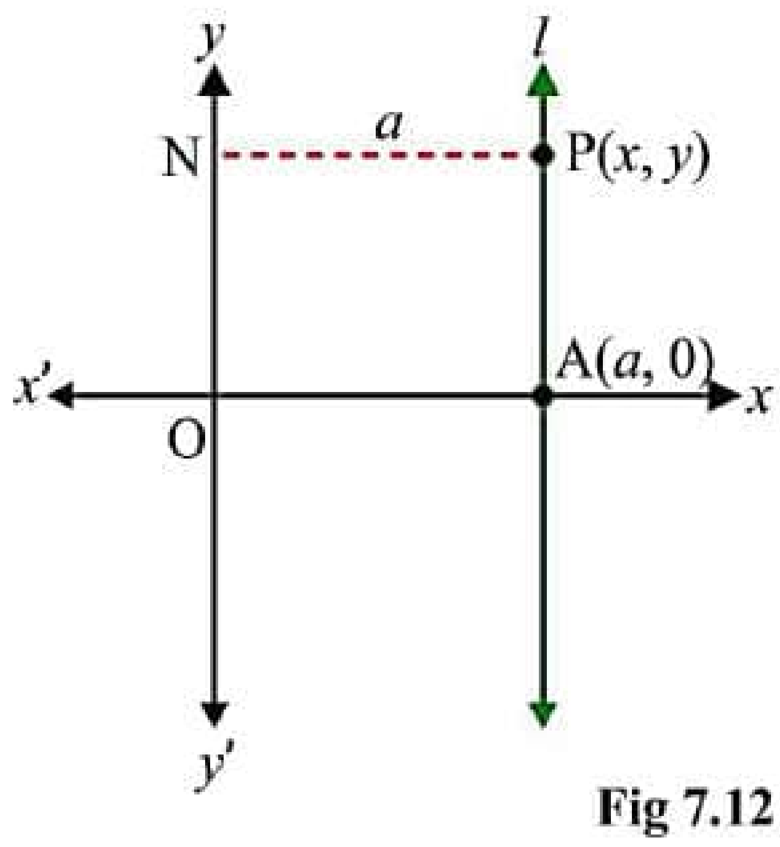


Fig 7.12

Let $P(x,y)$ be any point on l, draw $\overline{PN}$ perpendicular to y-axis.

Then $|\overline{NP}| = |\overline{OA}| = a$

i.e., $x = a$.

which is the equation of line parallel to y-axis.

If a is positive then line l is to the right of y-axis and if a is negative then line l is to the left of y-axis.

- **Line parallel to x-axis at a distance b from it**

Let l be a line parallel to x-axis is at a distance b from it and cutting the axis of y at B so that $|\overline{OB}| = b$, as shown in the figure 7.13.

Let $P(x, y)$ be any point on l. Draw $\overline{PM}$ perpendicular to x-axis.

Then, $|\overline{MP}| = |\overline{OB}| = b$

i.e., $y = b$

which is equation of line parallel to y-axis.

If b is positive then line is above the x-axis and if b is negative then line is below the x-axis.

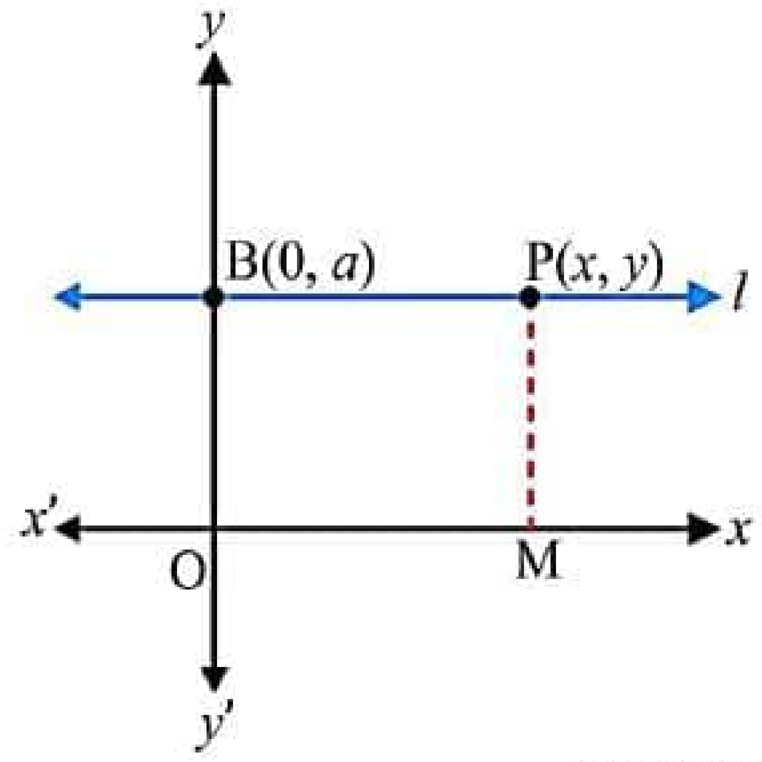


Fig 7.13

Corollary:

Since the axis of x is parallel to itself and at a distance zero from it, the equation of the x-axis is $y = 0$.

Since the axis of y is parallel to itself and at a distance 0 from it, the equation of the y-axis is $x = 0$.

Example 1. Find the equation of straight line parallel to the axis of x at a distance.

(i) 3 unit above it (ii) 5 unit below it

Solution:

(i) Since the line is parallel to x-axis and 3 unit above it, its equation $y = 3$.

(ii) Since the line is parallel to x-axis and 5 unit below it, its equation is $y = -5$.

Example 2. Find the equation of straight line parallel to the axis of y and a distance of

(i) 2 units to its right (ii) 7 units to its left

Solution:

(i) Since the line is parallel to the y-axis and 2 units to its right is $x = 2$.

(ii) Since the line is parallel to the x-axis and 7 units to its left is $x = -7$.

7.4 Standard Form of Equation of a Straight Line

7.4.1 Define intercepts of a straight line. Derive equation of a straight line in

- slope-intercept form
- point-slope form
- two-point forms
- intercepts form
- symmetric form
- normal form

- **x-intercept of a straight line**

When a straight line cuts the x-axis at a point $A(a, 0)$, then a is called x-intercept of the line, as shown in the figure 7.14.

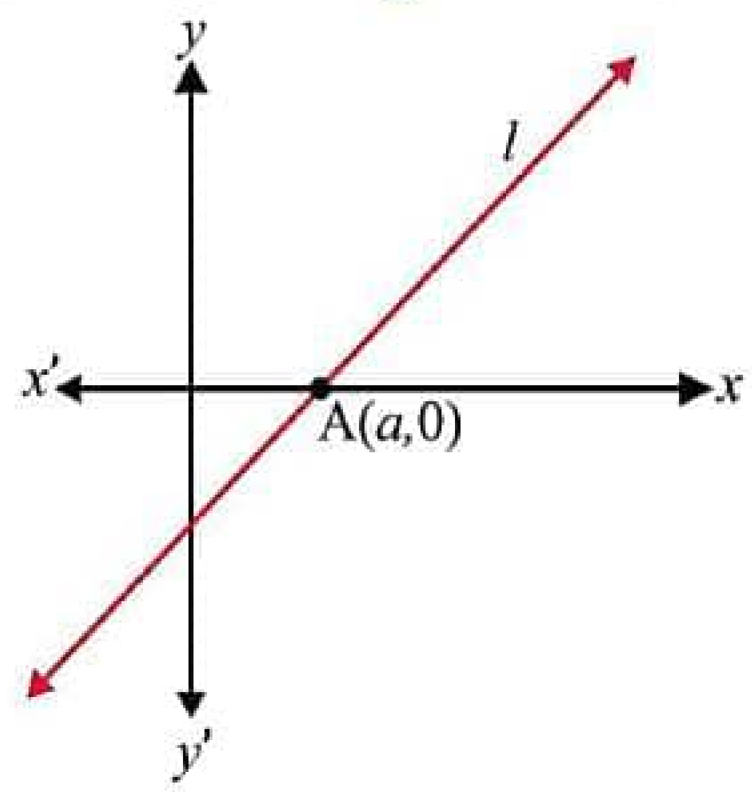


Fig 7.14

y-intercept of a straight line

When a straight line cuts the y-axis at a point $B(0, b)$ then b is called y-intercept of the line, as shown in the figure 7.15.

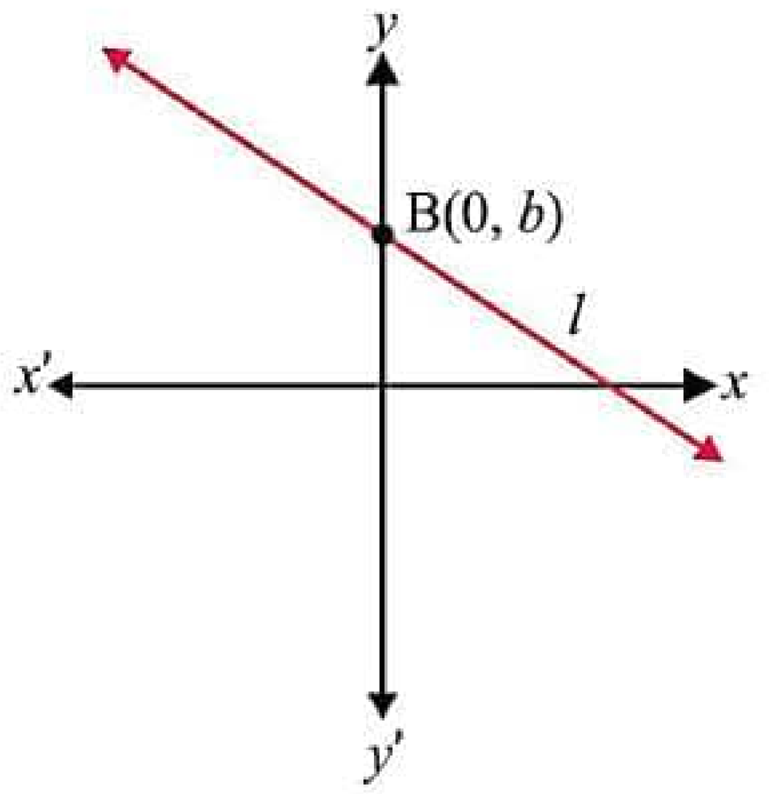


Fig 7.15

Example: If a line cuts the coordinates axes at $(3, 0)$ and $(0, -8)$ respectively. Find the x and y-intercept of the line.

Solution: Here the line cuts the x-axis at $(3, 0)$ then x-intercept of the line is 3. Similarly, the line cuts the y-axis at $(0, -8)$, then y-intercept of the line is -8.

Slope intercept form of straight line

Let the line l intersects the y-axis at P and θ is its inclination and $P(0, c)$ and $Q(x, y)$ be any two points on the line $\overline{AB}$ as shown in Fig. 7.16. Then the slope of the line as discussed in section 7.2 (i) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here,

$$m = \frac{y - c}{x - 0}$$

$$mx = y - c$$

$$y = mx + c$$

which is the required equation of the straight line with slope and intercept form, where m is the slope of the line and c is the y-intercept.

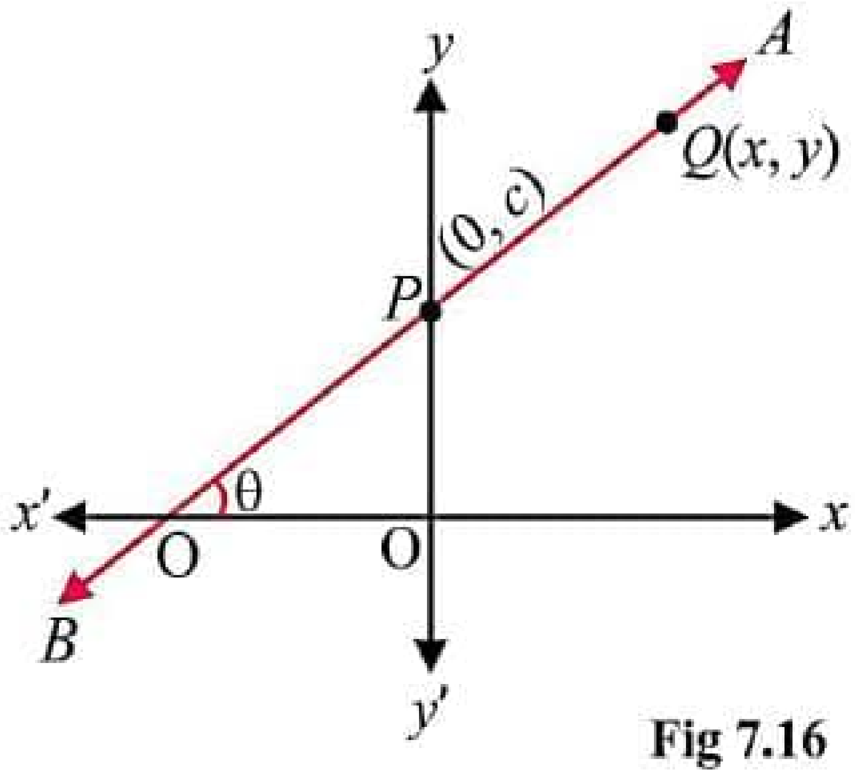


Fig 7.16

Example: Find the equation of a straight line whose slope is 3 which intersects the y-axis at $(0, 5)$.

Solution: We have, $m = 3$ and $c = 5$

The equation of a line in slope-intercept form is: $y = mx + c$.

So, the required equation is: $y = 3x + 5$

Note:

(i) If the slope or gradient i.e., $m = 0$ and y-intercept i.e., $c \neq 0$, then equation $y = mx + c \Rightarrow y = 0x + b \Rightarrow y = b$, which represents the equation of a line parallel to x-axis.

(ii) When slope and y-intercept is zero (i.e., $m = 0$ and $c = 0$) then equation $y = mx + c \Rightarrow y = 0x + 0 \Rightarrow y = 0$, which represents the equation of x-axis.

- **Point-slope form**

Consider a straight line l in the cartesian plane with slope m and a fixed point $Q(x_1, y_1)$ that lies on the line. Let $P(x, y)$ be another point on the line (Fig. 7.17).

Since the two points lie on the same line with slope m then;

$$m = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \quad (y - y_1) = m(x - x_1)$$

It is called point-slope form of equation of straight line that contains a fixed point $Q(x_1, y_1)$ and slope m.

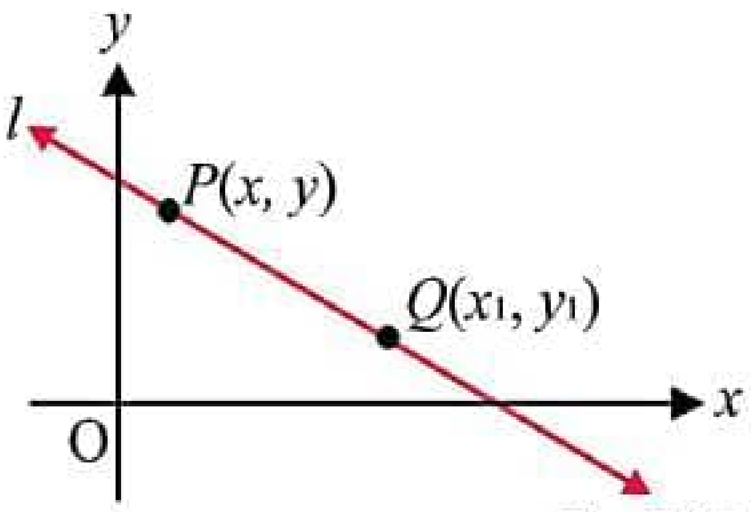


Fig 7.17

Corollary: If the line passes through the origin, i.e., if $x_1 = 0$ and $y_1 = 0$, then the equation of line is $y = mx$.

Example: Find the equation of straight line with slope -2 and passing through (2,6).

Solution: Here $m = -2$, $(x_1, y_1) = (2, 6)$. Using the point-slope formula we get;

$$(y - 6) = -2(x - 2)$$
$$y - 6 = -2x + 4$$
$$y + 2x - 10 = 0$$

which is required equation of straight line.

- **two-point form,**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points on line l_1. Let $P(x, y)$ be any any point on the line l.

From the (Fig. 7.18), we can say that the three points A, P and B are collinear. It shows that the slope of $\overline{AP}$ = slope of $\overline{AB}$.

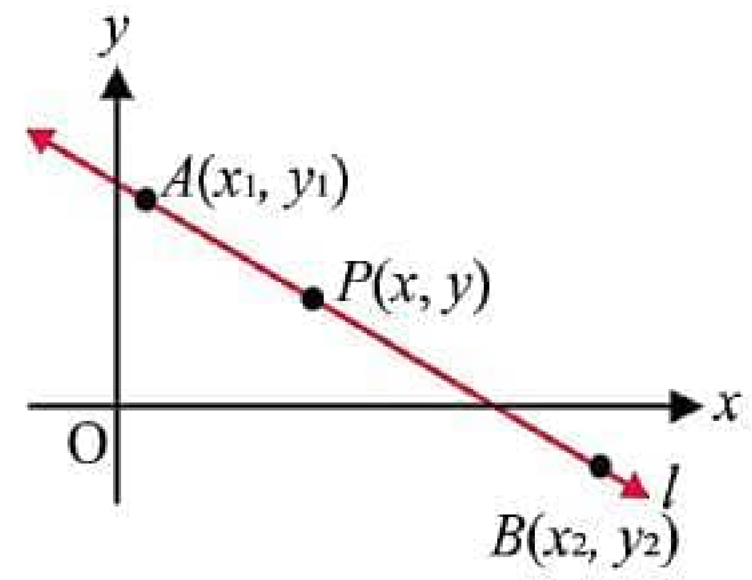


Fig 7.18

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the equation of a line in two-point form.

Corollary 1: The above equation can also be written in the form $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Corollary 2: Another way of writing the two-point form is;

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

Corollary 3: If the line passes through one point and origin, i.e. if $x_2 = 0, y_2 = 0$ then the equation of the line is;

$$\frac{y}{y_1} = \frac{x}{x_1}$$

Example 1. Find the equation of a line passing through the points $(-1, 2)$ and $(3, 5)$.

Solution: Let the given points be: $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (3, 5)$. Then using;

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

We get

$$y - 2 = \frac{(5 - 2)}{(3 + 1)}(x + 1)$$

$$3x - 4y + 11 = 0$$

This is the required equation of line passing through two points, $(-1, 2)$ and $(3, 5)$

Example 2. Find the equation of the line passing through the points $(1, -3)$ and $(5,7)$.

Solution: Using the Corollary (1), the required equation is;

$$\begin{vmatrix} x & y & 1 \\ 1 & -3 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 0$$

After simplifying the determinant, we get;

$$(-3 - 7)x - (1 - 5)y + (7 + 15) = 0$$

$$-10x + 4y + 22 = 0$$

or $$5x - 2y - 11 = 0$$

- **Intercepts form**

The intercepts form of the equation of the line can be derived from the two-point form of line. Let l is a line which passes through two points $(a, 0)$ and $(0, b)$ where a and b are the x and y intercepts of l respectively as shown in Fig. 7.19.

Thus, by two-point form of equation

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

We will get;

$$\frac{(y - 0)}{(x - a)} = \frac{(b - 0)}{(0 - a)}$$

$$y = \frac{-b}{a}(x - a)$$

$$y = \frac{-bx}{a} + b$$

$$y = b\left(\frac{-bx}{a} + 1\right)$$

$$\frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

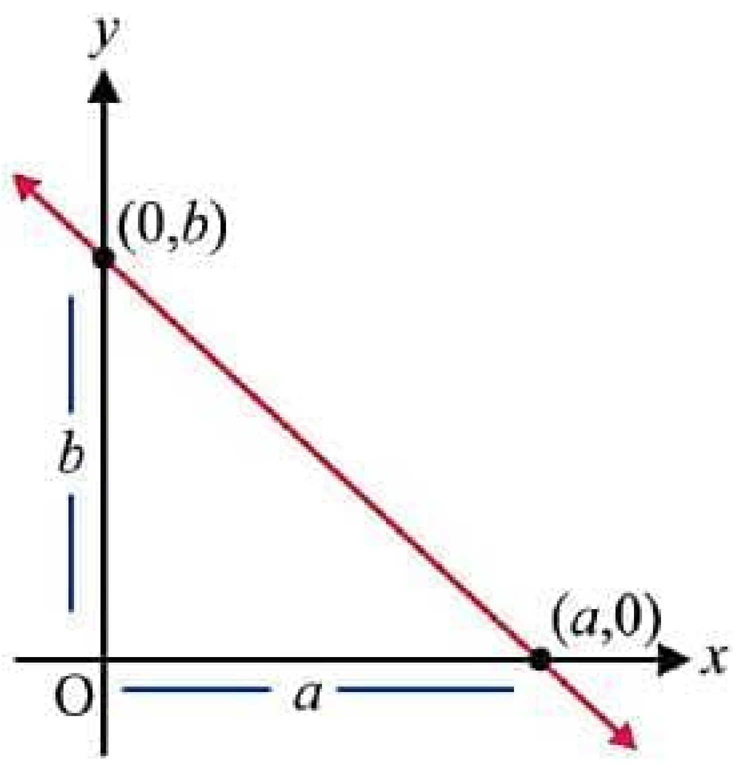


Fig 7.19

which is the required equation of straight line in intercepts form. Where a and b are x and y intercept respectively.

Corollary 1: The equation $\frac{x}{a}+\frac{y}{b}=1$ may be written in the form $lx+my=1$, where $l=\frac{1}{a}$ and $m=\frac{1}{b}$ and l, m are reciprocals of the intercepts on the axes.

Corollary 2: The equation of the straight-line which has equal intercept (say a) is $x+y=a$.

Example 1. Find the equation of the straight line which makes intercepts $\frac{1}{5}$ and $\frac{1}{7}$ on the axes respectively.

Solution: Here $a=\frac{1}{5}$ and $b=\frac{1}{7}$. Thus, the required equation is;

$$\frac{x}{\frac{1}{5}}+\frac{y}{\frac{1}{7}}=1$$

i.e., $5x+7x=1$

or $5x+7y-1=0$

- **Symmetric form,**

Let θ is the inclination of a straight-line l passing through the point $P(x,y)$. Consider another point $Q(x_1,y_1)$. Now using the slope formula;

$$m=\tan\theta=\frac{y-y_1}{x-x_1}$$

As we know that $\tan\theta=\frac{\sin\theta}{\cos\theta}$, then the above formula becomes;

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{y-y_1}{x-x_1}$$

$$\frac{\sin\theta}{\cos\theta}=\frac{y-y_1}{x-x_1}$$

or

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}$$

This is called the symmetric form of an equation of a straight line with inclination θ and passing through (x_1,y_1).

Example 7. Find the symmetric form equation of a straight line with inclination 45° and passing through the point $(2,\sqrt{2})$.

Solution: Here an inclination is $\theta=45°$ and point $(x_1,y_1)=(2,\sqrt{2})$. Using the equation of line in its symmetric form;

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}$$

Substitute the above values in the formula to get the symmetric form equation of a straight line;

$$\frac{x-2}{\cos 45^\circ} = \frac{y-\sqrt{2}}{\sin 45^\circ}$$

$$\frac{x-2}{\left(\frac{1}{\sqrt{2}}\right)} = \frac{y-\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)}$$

$\Rightarrow \quad y - x + 2 - \sqrt{2} = 0$ is __________ equation.

- **Normal form or perpendicular form**

Let $P(x, y)$ be any point on the straight-line l. The line intersects the coordinate axes at points A and B respectively. The $|\overline{OA}|$ and $|\overline{OB}|$ become its x-intercept and y-intercept respectively as shown in the Fig. 7.20.

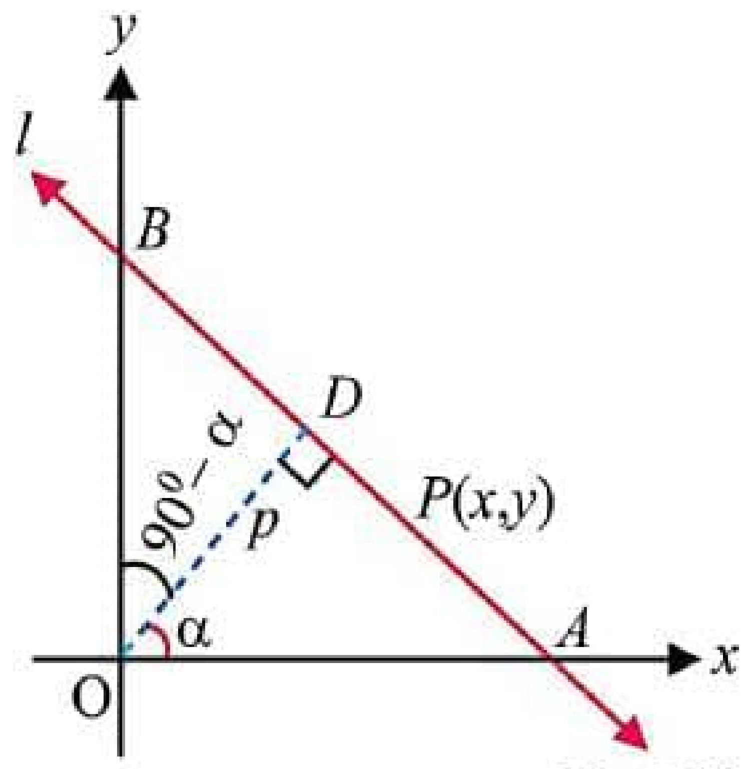


Fig 7.20

Now using intercepts form of equation of straight line, we have

$$\frac{x}{|\overline{OA}|} + \frac{y}{|\overline{OB}|} = 1 \qquad \text{... (i)}$$

Let p be the length of the normal drawn from the origin to the line, which subtends an angle α with the positive direction of x-axis. If D is foot of perpendicular drawn from O then consider the triangle ODA as given in Fig. 7.20. Using the trigonometric ratios; we get;

$$\frac{|\overline{OD}|}{|\overline{OA}|} = \cos\alpha$$

$$\Rightarrow \quad \frac{p}{|\overline{OA}|} = \cos\alpha$$

$$\Rightarrow \quad |\overline{OA}| = \frac{p}{\cos\alpha} \qquad (\because |\overline{OD}| = p)$$

Similarly, ODB is a right-angle triangle, then;

$$\frac{|\overline{OD}|}{|\overline{OB}|} = \cos(90^\circ - \alpha)$$

$$\Rightarrow \quad \frac{p}{|\overline{OB}|} = \sin\alpha$$

$$\Rightarrow \quad |\overline{OB}| = \frac{p}{\sin\alpha}$$

Now substituting the values of $|\overline{OA}|$ and $|\overline{OB}|$ in equation (i) we get;

$$\frac{x}{\frac{p}{\cos\alpha}} + \frac{y}{\frac{p}{\sin\alpha}} = 1$$

$$\Rightarrow \quad \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow \quad x\cos\alpha + y\sin\alpha = p$$

where p is always kept positive and α is measured in counter-clockwise direction $(0 < \alpha < 2\pi)$.

This equation is called normal or perpendicular form of equation of line.

Example 8. Find the equation of the straight line which is at a of distance 9 units from the origin and the perpendicular from the origin to the line makes an angle 30◦ with the positive direction of x-axis.

Solution: Here $p = 9$ and $\alpha = 30°$, Using the Normal form of equation of straight line, we have

$$x\cos 30° + y\sin 30° = 9$$

$$x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) = 9$$

$$\frac{x\sqrt{3}}{2} + \frac{y}{2} = 9$$

$$x\sqrt{3} + y = 18$$

which is the required equation of line.

(ii) Show that a linear equation in two variables represents a straight line.

A linear equation in two variables x and y is the equation of the form

$$ax + by + c = 0 \qquad \ldots(i)$$

where a, b and c are real numbers (constants). Also, a and b are not both zero.

Theorem: Every linear equation in two variables represents a straight line.

Linear equation into two variables is $ax + by + c = 0$, both a and b are not 0.

Proof: Case-I when $b = 0$.

In this case $a \neq 0$. Thus, the equation (i) reduces to

$$ax + c = 0$$

$$x = \frac{-c}{a} \qquad \ldots(ii)$$

which is the equation of straight line and parallel to y-axis.

Case-II when $a = 0$.

In this case $b \neq 0$. Thus, the equation (i) reduces to

$$by + c = 0$$

$$y = \frac{-c}{b} \qquad \ldots(iii)$$

which is the equation of straight line and parallel to x-axis.

Case-III When $a \neq 0$ and $b \neq 0$. In this the equation (i) reduces to

$$ax + by + c = 0$$

$$y = \frac{-a}{b}x + \frac{-c}{b}$$

$$y = mx + c' \qquad \text{... (iv)}$$

where $m = \frac{-a}{b}$ and $c' = \frac{-c}{b}$. Thus (iv) is also equation of straight line in slope intercept form. Hence in all cases a linear equation in two variables represents a straight line.

(iii) Reduce the general form of the equation of a straight line to the other standard forms.

- **Reduce the general equation $ax + by + c = 0$ into slope intercept form**

  The general equation of straight line is

  $$ax + by + c = 0 \qquad \text{...(i)}$$

  Now adding $-ax - c$ on both sides of equation (i) we get;

  $$ax + by + c - ax - c = -ax - c$$

  $$by = -ax - c$$

  $$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$

  which is the slope intercept form of line, where $\frac{-a}{b}$ is the slope and $\frac{-c}{b}$ is the y-intercept from the line.

- **Reduce the general equation $ax + by + c = 0$ into intercept form.**

  The general equation of straight line is

  $$ax + by + c = 0 \qquad \text{...(i)}$$

  If $a \neq 0, b \neq 0, c \neq 0$ then from the equation (i) we get,

  $$ax + by = -c$$

  $$\frac{ax}{c} + \frac{by}{c} = \frac{-c}{c} \qquad \text{(Dividing both sides by } -c\text{)}$$

  $$\frac{x}{\frac{c}{a}} + \frac{y}{\frac{c}{b}} = -1$$

  $$\Rightarrow \quad \frac{x}{\left(\frac{-c}{a}\right)} + \frac{y}{\left(\frac{-c}{b}\right)} = 1$$

  which is the required intercept form of equation of line, where $\frac{-c}{a}$ is the x-intercept and $\frac{-c}{b}$ is the y-intercept.

- **Reduce the general equation $ax + by + c = 0$ into Normal form.**

  The general equation of straight line is

  $$ax + by + c = 0 \qquad \text{...(i)}$$

Let the normal equation of line is

$$x\cos\alpha + y\sin\alpha - p = 0 \qquad \text{where } p > 0 \qquad \ldots \text{(ii)}$$

By comparing equation (i) and (ii) as both the equations are identical, we get;

$$\Rightarrow \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c} = k$$

Thus $\cos\alpha = ak$ and $\sin\alpha = bk, p = -ck$.

Squaring and adding we get;

$$\cos\alpha^2 + \sin\alpha^2 = a^2k^2 + b^2k^2$$

$$\Rightarrow \quad a^2k^2 + b^2k^2 = 1$$

$$\Rightarrow \quad k = \pm\frac{1}{\sqrt{a^2+b^2}}$$

Since p is always positive therefore select k as positive. Thus, the equation (ii) becomes;

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = -\frac{c}{\sqrt{a^2+b^2}} \qquad \ldots \text{(iii)}$$

which is the required normal form of equation of line.

Exercise 7.3

1. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(3, -4)$.
2. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(5, 2)$.
3. Write the equation of the straight lines parallel to x-axis which is at a distance of 5 units from above the x-axis.
4. Find the equation of a line parallel to y-axis which is at a distance of 6 units on its left.
5. Find the equations of the straight line determined by each of the following set of conditions.

   (i) through $(5, -2)$ with the slope 4

   (ii) through $(-1, -4)$ with the slope $\frac{-2}{3}$

   (iii) through $\left(\frac{-1}{4}, \frac{3}{4}\right)$ with slope $\frac{2}{5}$

   (iv) through $(0, b)$ with the slope m.

   (v) through the points $(7, -3)$ and $(-4, 1)$

   (vi) through the points $(5, -5)$ and $(-3, 1)$

   (vii) through $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

(viii) y-intercept = 3; slope = 2

(ix) y-intercept = -2; slope = $\frac{-2}{3}$

(x) y-intercept = -5; slope = $\frac{1}{2}$

(xi) y-intercept = 0; slope = 0

(xii) x-intercept = 4; y-intercept = 3

(xiii) x-intercept = -2; y-intercept = 5

(xiv) x-intercept = -5; y-intercept = -1

(xv) the perpendicular from the origin to the line, $p = 3$ units and it makes an angle $\alpha = 60°$ with x-axis.

(xvi) $p = \frac{3}{2}, \alpha = 150°$

6. Reduce the equation $3x + 4y - 12 = 0$ to the

(i) slope-intercept form (ii) two-intercept form

(iii) Normal or Perpendicular form

7. Find the equations of the sides of the triangle whose vertices are $(1,4), (2,-3)$ and $(-1,-2)$.

8. Find the equation of the perpendicular bisector of the segment joining $(-1,2)$ and $(9,12)$.

9. The x-intercept of a line is the reciprocal of its y-intercept and line passes through $(2,-1)$. Find its equation.

10. Find the equation of the line which passes through $(-2,-4)$ and the sum of its intercepts equal to 3.

11. Find the equation of the line which passes through $(5,6)$ and the y-intercept is twice that of the x-intercept.

12. Find an equation of the line through $(11,-5)$ and parallel to a line having slope $\frac{3}{2}$.

13. Find an equation of the line through $(-4,-6)$ and perpendicular to the line having slope $\frac{-3}{2}$.

7.5 Distance of a Point from a Line

We know that using the distance formula we can find the distance between the two points that could be a distance between two objects, a distance between two houses, etc. Similarly, the distance from a point can be measured from a line as well. Thus, the perpendicular distance of a point from a line is the shortest distance between the point and the line.

7.5.1 Recognize a point with respect to position of a line

As we know that a line divides a plane into two regions such that every point of the plane not on the line lies in one of the regions. Thus, if a line is not parallel to y-axis, every point of the plane not on the line is either above the line or below the line.

Theorem 1: Let l denote the line $ax+by+c=0$ with $b>0$.

If $P(x_1,y_1)$ is a point above the line l, then $ax_1+by_1+c>0$.

If $P(x_2,y_2)$ is a point below the line l, then $ax_1+by_1+c<0$.

Proof: Through P_1 draw a line l' parallel to y-axis, intersecting the line l at a point M (Fig. 7.21).

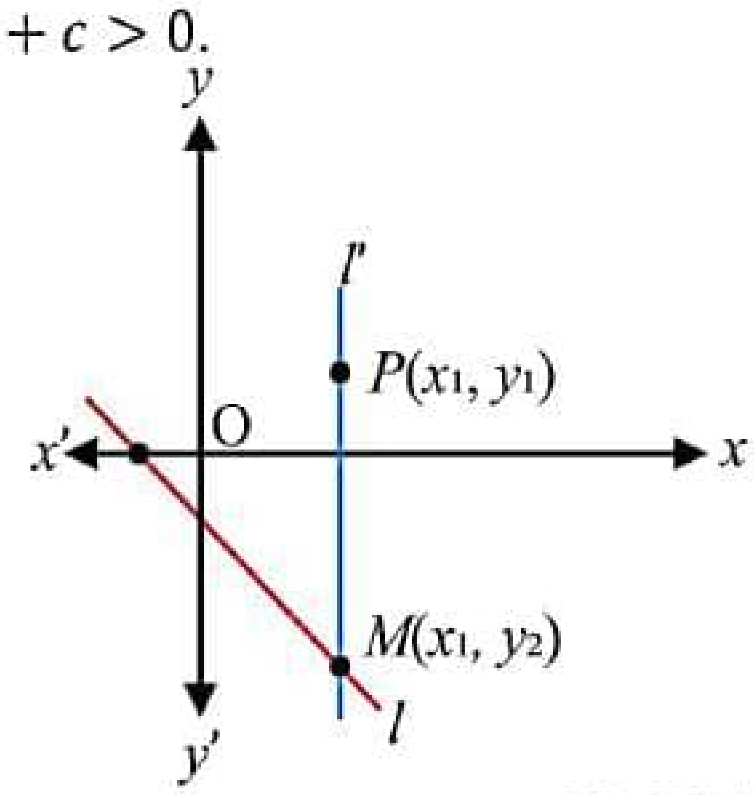


Fig 7.21

Now the abscissa of the M is same as the abscissa of P_1, viz, x_1. Let us denote the ordinate of M by y_2, so that M is the point (x_1,y_2).

If P_1 is a point lies above the line then $y_1>y_2$. Since $b>0$ we have $by_1>by_2$. Adding ax_1+c to both sides of this inequality we have

$$ax_1+by_1+c>ax_1+by_2+c$$

Since $M(x_1,y_2)$ lies on the line l, $ax_1+by_2+c=0$.

Thus, $ax_1+by_1+c>0$

If $P_1(x_1,y_1)$ is a point lies below the line then $y_1<y_2$ and in a similar way we can show that

$$ax_1+by_1+c<0$$

The following is the converse of the theorem 1.

Theorem 2: Let l denote the line $ax+by+c=0$ with $b>0$.

(i) If x_1 and y_1 are real numbers such that $ax_1+by_1+c>0$ then the point $P_1(x_1,y_1)$ lies above the line l.

(ii) If x_1 and y_1 are real numbers such that $ax_1+by_1+c<0$ then the point $P_1(x_1,y_1)$ lies below the line l.

Note: The above two theorems should be used after the equation of the given line has been put in the general form; i.e., $ax_1+by_1+c=0$ with $b>0$

In case $b=0$ the line would be parallel to y-axis then the question of a point below or above does not arise.

Example: Find whether each of the points $(-8,-3),(10,-5),(-35,9)$ is above or below the line $2x-3y+4=0$.

Solution: First of all, we write the equation of straight line in the form in which coefficient of y is positive. Thus,

$$-2x+3y-4=0$$

For the point $(-8,-3)$, we have,

$$-2(-8) + 3(-3) - 4 = 3 > 0$$

Hence $(-8,-3)$ lies above the given line. Again, for the point $(10,-5)$, we have,

$$-2(10) + 3(-5) - 4 = -39 < 0$$

Hence $(10,-5)$ lies below the given line. Again, for the point $(-35,9)$, we have,

$$-2(-35) + 3(9) - 4 = 93 > 0$$

Hence $(-35,9)$ lies above the given line.

7.5.2 Find the perpendicular distance from a point to the given straight line.

Consider a line $l: ax + by + c = 0$ and a point $P(x_1, y_1)$ not on l.

To find perpendicular distance of P from l. Through P, draw a line l' perpendicular to l cutting l at Q. (Fig. 7.22)

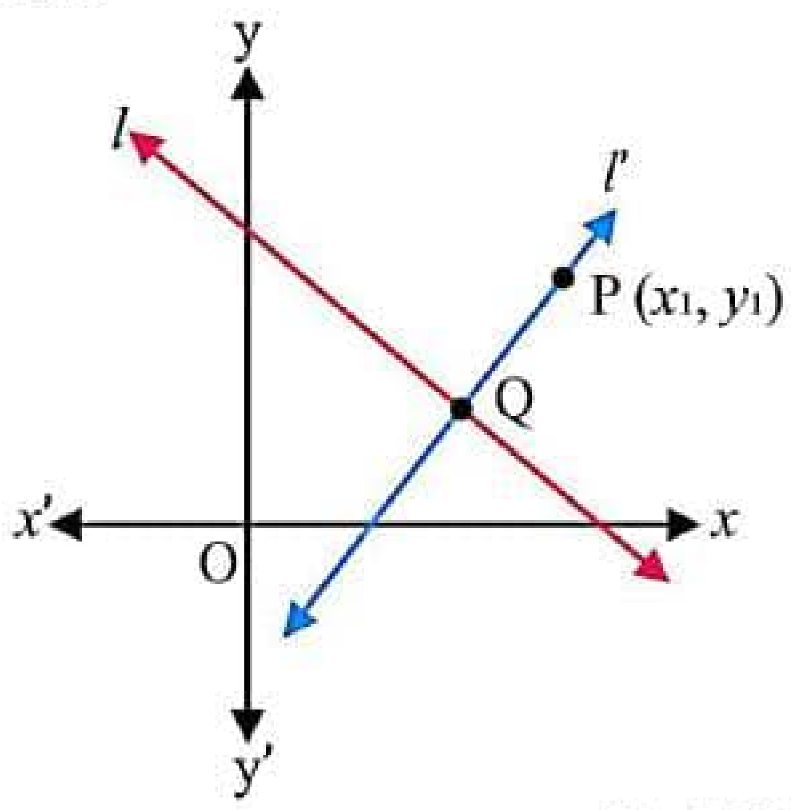


Fig 7.22

Slope of $l = -\frac{a}{b} \Rightarrow$ slope of $l' = \frac{b}{a}$

Equation of l' is $y - y_1 = \frac{b}{a}(x - x_1)$

$\Rightarrow \quad bx - ay + ay_1 - bx_1 = 0$

To find Q, we solve equation of l and l', we get

$$\frac{x}{b(ay_1 - bx_1) + ac} = \frac{y}{bc - a(-bx_1 + ay_1)} = \frac{1}{-a^2 - b^2}$$

$$\Rightarrow \quad \frac{x}{aby_1 - b^2x_1 + ac} = \frac{1}{-a^2 - b^2} \text{ and } \frac{y}{bc + abx_1 - a^2y_1} = \frac{1}{-a^2 - b^2}$$

$$\Rightarrow \quad x = \frac{aby_1 - b^2x_1 + ac}{-a^2 - b^2} \qquad \Rightarrow \quad y = \frac{bc + abx_1 - a^2y_1}{-a^2 - b^2}$$

So,

$$x = \frac{b^2x_1 - aby_1 - ac}{a^2 + b^2} \qquad y = \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}$$

Thus, the coordinates of Q are $\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}\right)$

By using distance formula,

$$d = |\overline{PQ}| = \sqrt{\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2} - x_1\right)^2 + \left(\frac{a^2y_1 - abx_1 - bc}{a^2 + b^2} - y_1\right)^2}$$

$$= \sqrt{\frac{(a^2x_1 + aby_1 + ac)^2}{(a^2 + b^2)^2} + \frac{(abx_1 + b^2y_1 + bc)^2}{(a^2 + b^2)^2}}$$

$$= \sqrt{\frac{a^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2} + \frac{b^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \sqrt{\frac{(ax_1 + by_1 + c)^2(a^2 + b^2)}{(a^2 + b^2)^2}}$$

$$d = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$\because$ d is always $+ve$

$\therefore$ $d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

In case $b = 0$, the same formula holds.

Example: Find the distance of the point $(-3, 5)$ from the line $4x - 3y - 26 = 0$.

Solution: Given equation of a line is: $4x - 3y - 26 = 0$ and the point $(x_1, y_1) = (-3, 5)$

Comparing these with the general forms,

$$a = 4, \quad b = -3, \quad c = -26$$

We know that the perpendicular distance (d) of a line $ax + by + c = 0$ from a point (x_1, y_1) is given by

$$d = \left[\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right]$$

After substituting the values, we will get;

$$d = \left|\frac{(4)(-3) + (-3)(5) - 26}{\sqrt{(4)^2 + (-3)^2}}\right|$$

$$d = \frac{53}{5}$$

7.5.3 Find the distance between two parallel lines

The distance between two parallel lines is equal to the perpendicular distance between the two lines. We know that the slopes of two parallel lines are the same; therefore, the equation of two parallel lines can be given as:

$$y = mx + c_1 \qquad \text{...(i)}$$

$$y = mx + c_2 \qquad \text{...(ii)}$$

The point $A\left(-\frac{c}{m}, 0\right)$ is the intersection point of the second line and the x-axis (Fig. 7.23). The perpendicular distance from A to l_1 will be the required perpendicular distance between

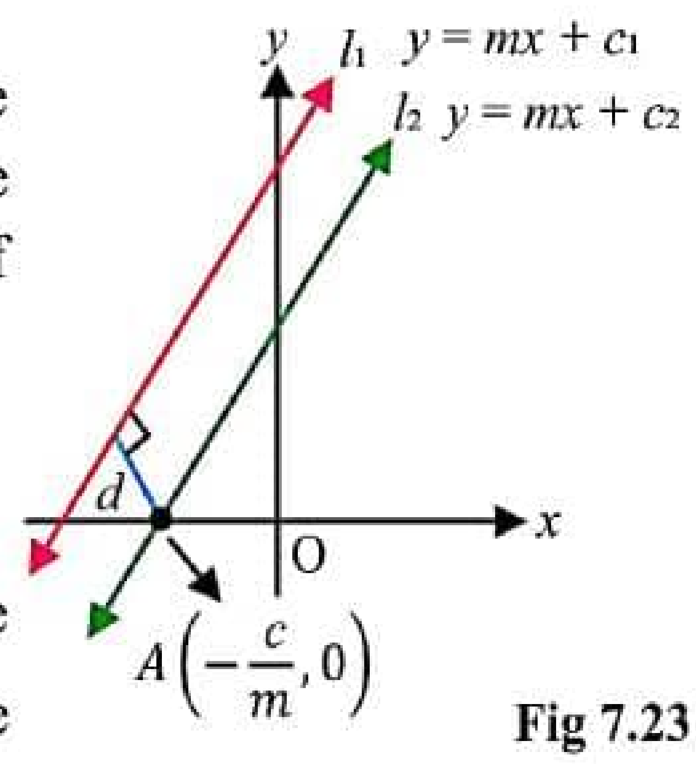


Fig 7.23

two parallel lines. The distance between the point A and the line $y = mx + c_2$ can be given by using the formula:

$$d = \left[\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right]$$

$$d = \left[\frac{(-m)\left(\frac{-c_1}{m}\right) - c_2}{\sqrt{(1 + m^2)}}\right]$$

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

Thus, we can conclude that the perpendicular distance between two parallel lines is given by:

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

Example: Find the distance between two parallel lines $3x + 4y = 9$ and $6x + 8y = 15$.

Solution: Given equations of lines are:

$$3x + 4y = 9 \qquad \ldots\text{(i)}$$

$$6x + 8y = 15$$

or $$3x + 4y = \frac{15}{2} \qquad \ldots\text{(ii)}$$

Now, by comparing with the general equations of straight lines we get;

$$a = 3, b = 4, c_1 = -9 \text{ and } c_2 = \frac{-15}{2}$$

Thus, the required distance will be;

$$d = \frac{\left|(-9) - \left(\frac{-15}{2}\right)\right|}{\sqrt{(3)^2 + (4)^2}}$$

$$d = \frac{3}{5}$$

That is the required distance between two lines.

Exercise 7.4

1. Determine whether each of the specified points is above or below the given straight line:
   (i) $3x + 11y - 44 = 0$, $(10, 1)$, $(-4, 6)$ and $(5, 3)$
   (ii) $10x - 12y + 17 = 0$, $(-20, -15)$, $(5, 5)$ and $(100, 84)$
   (iii) $29x - 17y + 31 = 0$, $(0, 2)$, $(-3, -3)$ and $(20, 30)$
2. In each of the following, find the perpendicular distance from the point to the line;
   (i) $15x - 8y - 5 = 0$, $(2, 1)$
   (ii) $3x - 4y + 5 = 0$, $(4, -3)$

(iii) $3x + 4y + 10 = 0, (3, -2)$

(iv) $2x - 7y + 1 = 0, (7, 4)$

(v) $5x + 12y - 16 = 0, (3, -1)$

3. Find the distance between the parallel lines;

(i) $5x - 12y + 10 = 0, 5x - 12y - 16 = 0$

(ii) $x + y - 2 = 0, 2x + 2y - 1 = 0$

(iii) $4x - 3y + 12 = 0, 4x - 3y - 12 = 0$

7.6 Angle Between Lines

7.6.1 Find the angle between two coplanar intersecting straight lines

Let the equations of the straight lines l_1 and l_2 are $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively intersect at a point P and make angles θ_1 and θ_2 respectively with the positive direction of x-axis as shown in Fig. 7.24.

Let $\angle APC = \theta$ is positive angle from l_1 to l_2.

Clearly, the slope of the line l_1 and l_2 are m_1 and m_2 respectively.

Then, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Now, from the elementary geometry

$$\theta_2 = \theta + \theta_1 \qquad \Rightarrow \theta = \theta_2 - \theta_1$$

Now, $\tan\theta = \tan(\theta_2 - \theta_1) = \dfrac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1 \tan\theta_2}$

Thus, substituting the values of $\tan\theta_1$ and $\tan\theta_2$ for m_1 and m_2 respectively, we have;

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

It should be noted that the value of $\tan\theta$ in this equation will be positive if θ is acute and negative if θ is obtuse.

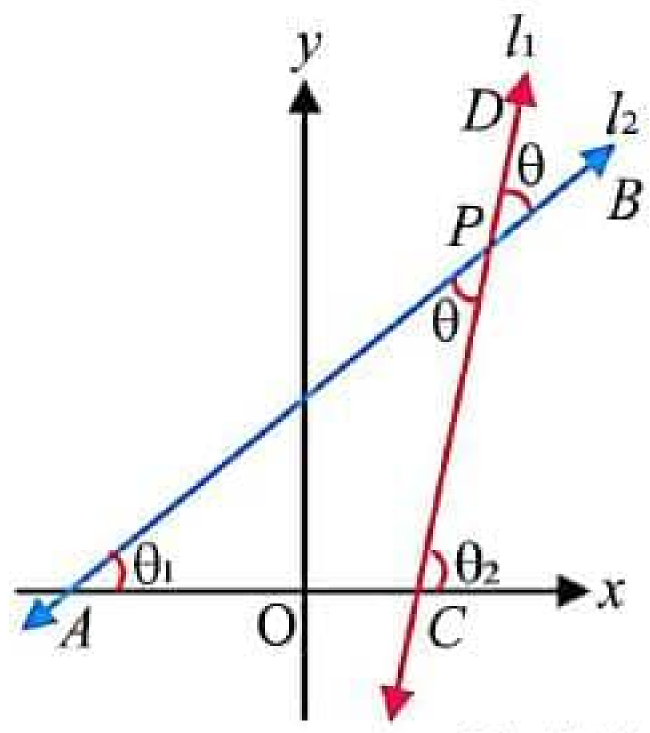


Fig 7.24

- The angle between two lines having equations $l_1: a_1x + b_1y + c_1 = 0$ and $l_2: a_2x + b_2y + c_2 = 0$ is $\theta = \tan^{-1}\dfrac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$.
- If the lines l_1 and l_2 are perpendicular, then $\theta = 90°$ or $\dfrac{\pi}{2}$. Thus, we have,

$$1 + m_1m_2 = \frac{m_2 - m_1}{\tan\theta}$$

$$\Rightarrow \quad 1 + m_1m_2 = \frac{m_2 - m_1}{tan\,90°}$$

$$\Rightarrow \quad 1 + m_1m_2 = \frac{m_2 - m_1}{\infty}$$

$\Rightarrow \quad 1 + m_1 m_2 = 0$

$\Rightarrow \quad m_1 m_2 = -1$ or $a_1 a_2 + b_1 b_2 = 0$

This is the condition for two lines to be perpendicular.

- If the lines l_1 and l_2 are parallel, then $\theta = 0°$ or π radian. Thus, we have,

$$a_2 b_1 - a_1 b_2 = 0$$

- If two linear equations have the same x and y coefficients, the lines represented by them are parallel.
- If the coefficients of the later of the two linear equations are those of the former reversed in order and with the sign of one coefficient changed, the lines represented by them are perpendicular.

For example, line $ax + by + c = 0$ is respectively parallel to $ax + by + c_1 = 0$ and perpendicular to $ax - by + c_2 = 0$.

Example 1. If $A(-2, 1), B\ (2, 3)$ and $C\ (-2, -4)$ are three points, find the acute angle between the straight lines AB and BC.

Solution: Let the slope of the line AB and BC are m_1 and m_2 respectively.

Then, $m_1 = \frac{3-1}{2-(-2)} = \frac{1}{2}$ and $m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$

Let θ be the angle between AB and BC. Then,

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} = \pm\frac{2}{3}$$

$\theta = \tan^{-1}\left(\frac{2}{3}\right)$ is the required acute angle.

Example 2. Find the acute angle between the lines $7x - 4y = 0$ and $3x - 11y + 5 = 0$.

Solution:

Method 1: First we need to find the slope of both the lines.

Thus, $7x - 4y = 0$ the slope of the line is $\frac{7}{4}$.

Again, $3x - 11y + 5 = 0$, the slope of the line is $\frac{3}{11}$.

Using the formula, we have; $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

$$\theta = \tan^{-1}\left(\frac{\frac{7}{4} - \frac{3}{11}}{1 + \left(\frac{3}{11}\right)\left(\frac{7}{4}\right)}\right) = \tan^{-1}(1)$$

$\because\ \theta$ is acute $\therefore\ \theta = 45°$.

Method 2: The given two equations of the lines are $7x - 4y = 0$ and $3x - 11y + 5 = 0$.

Here we have $a_1 = 7, b_1 = -4, a_2 = 3$ and $b_2 = -11$.

The angle between the two lines can be calculated using the formula

$$\tan\theta = \frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$$

$$= \frac{3(-4) - (7)(-11)}{(3)(7) + (-4)(-11)}$$

$$= \frac{-12 + 77}{21 + 44} = \frac{65}{65} = 1$$

$$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45°$$

7.6.2 Find the equation of family of lines passing through the point of intersection of two given lines.

A family of lines is a set of lines having one or two factors in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the y-intercept is the same.

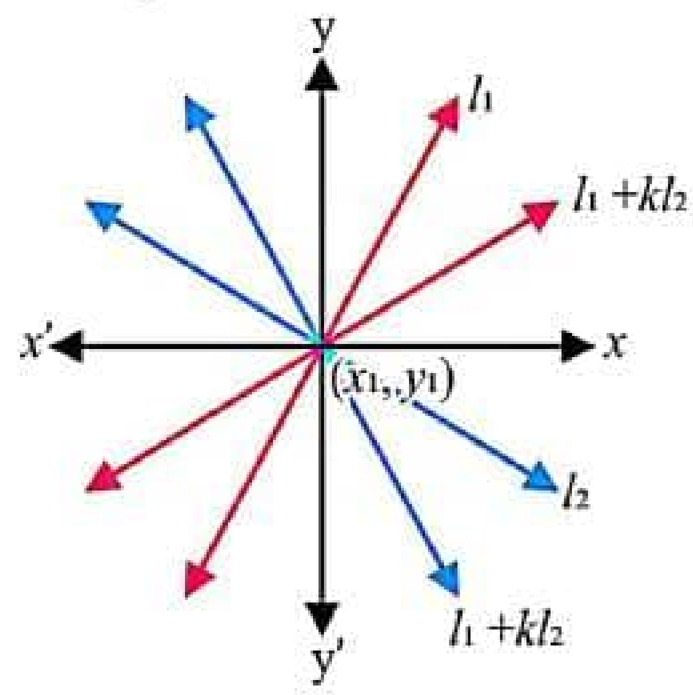


Fig 7.25

Consider the two straight lines;

$a_1x + b_1y + c_1 = 0$ ...(i)

$a_2x + b_2y + c_2 = 0$ ...(ii)

For any nonzero constant k, the equation of the form

$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$ ...(iii)

being linear in x and y is an equation of a straight line.

If (x_1, y_1) is the point of intersection of line (i) and (ii) then it must satisfy the both equation (i) and (ii);

$a_1x_1 + b_1y_1 + c_1 = 0$ ...(iv)

$a_2x_1 + b_2y_1 + c_2 = 0$ ...(v)

Next, we check whether the point (x_1, y_1) lies on (iii) or not. For this, we replace x by x_1 and y by y_1 in equation (iii), we will get,

$a_1x_1 + b_1y_1 + c_1 + k(a_2x_1 + b_2y_1 + c_2) = 0$ ...(vi)

Using equation (iv) and (v) in Equation (vi), we will get;

$$0 + k(0) = 0$$

This shows that equation (vi) is true for all k and for $x = x_1, \ y = y_1$. Thus, the point (x_1, y_1) lies on (vi) for all k. Equation (vi) represents the equation of the line through the point of intersection of lines (i) and (ii). Since k is any real number, equation (vi) shows that there will be an infinite number of lines (Family of Lines) through the point of intersection of lines (i) and (iii).

Example: Find the equation of a line through the point (1,3) and the point of intersection of lines $2x - 3y + 4 = 0$ and $4x + y - 1 = 0$.

Solution: The family of the equation of a straight line through the point of intersection of lines $2x - 3y + 4 = 0$ and $4x + y - 1 = 0$ is given as

$$2x - 3y + 4 + k(4x + y - 1) = 0 \qquad \ldots\text{(i)}$$

Since the required line passes through the point (1,3), this point must satisfies the equation (i) i.e.

$$2(1) - 3(3) + 4 + k(4(1) + 3 - 1) = 0 - 3 + 6k = 0$$

$$k = 12$$

Substituting the value of k in the required equation of a straight line, we have

$$2x - 3y + 4 + 12(4x + y - 1) = 0$$

$8x - 5y + 7 = 0$ is the required equation of straight line.

7.6.3 Calculate angles of the triangle when the slopes of the sides are given

The angles of triangle will be calculated using the formula as discussed in section 7.6(i)

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of two lines.

Example: Find the angles of the given triangle, the slopes of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are $-3, 2$ and $\frac{1}{3}$ respectively.

Solution:

Let m_1 is the slope of $\overline{AB}$ i.e., $m_1 = -3$, m_2 is the slope of $\overline{BC}$ i.e., $m_2 = 2$ and m_3 is the slope of $\overline{AC}$ i.e., $m_3 = \frac{1}{3}$

Assume θ_1 is the positive angle from $\overline{AB}$ to $\overline{AC}$, θ_2 is the positive angle from $\overline{BC}$ to $\overline{AB}$ and θ_3 is the positive angle from $\overline{AC}$ to $\overline{BC}$.

Now,

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3}$$

$$\tan\theta_1 = \frac{(-3) - \left(\frac{1}{3}\right)}{1 + (-3)\left(\frac{1}{3}\right)}$$

$$\tan\theta_1 = \frac{-8}{0} \qquad \text{(undefined)}$$

$$\tan\theta_1 = -\infty$$

$$\boxed{\theta_1 = 90^\circ}$$

Similarly,

$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta_2 = \frac{(-3) - (2)}{1 + (-3)(2)}$$

$$\tan\theta_2 = \frac{-5}{-5}$$

$$\boxed{\theta_2 = 45°}$$

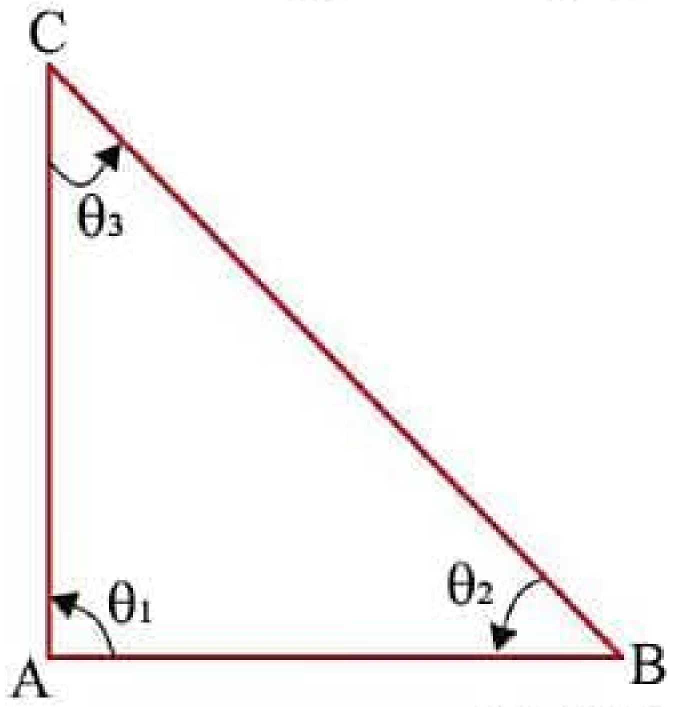


Fig 7.26

and

$$\theta_3 = 180° - \theta_1 - \theta_2$$

$$\theta_3 = 180° - 90° - 45°$$

$$\boxed{\theta_3 = 45°}$$

These interior angles of the triangles are 90°, 45° and 45°.

Exercise 7.5

1. What is the angle between two lines when they intersect at origin and one of the line passes through $(2, 3)$ and the other line passes through $(-3, 6)$?
2. Find the angle between the following two lines. $l_1: 4x - 3y = 8$ and $l_1: 2x + 5y = 4$.
3. Find the acute angle between $l_1: y = 3x + 1$ and $l_2: y = -4x + 3$.
4. Find the angle between two lines, one of which is the x-axis and the other line is $x - y + 4 = 0$, is?
5. Find the angle between the lines $2x - 3y + 7 = 0$ and $7x + 4y - 9 = 0$.
6. Find the equation of line through point (3, 2) and making angle 45° with the line $x - 2y = 3$.
7. Determine the measure of the acute angle between the straight-line $x - y + 4 = 0$ and the straight line passing through the points $(3, 2)$ and $(2, 4)$.
8. Find the equation of family of lines that pass through the point of intersection of $2x + 3y - 8 = 0$ and $x - y + 1 = 0$. Also find the point of intersection.
9. Find the equation of a line through the intersection of the lines;
   (i) $2x + 3y + 1 = 0, 3x - 4y = 5$ and passing through the point (2, 1).
   (ii) $x - 4y = 3, x + 2y = 9$ and passing through the origin.
   (iii) $3x + 2y = 8, 5x - 11y + 1 = 0$ and parallel to $6x + 13y = 25$.
   (iv) $2x - 3y + 4 = 0, 3x + 3y - 5 = 0$ and parallel to y-axis.
   (v) $5x - 6y = 1, 3x + 2y + 5 = 0$ and perpendicular to $5y - 3x = 11$.
   (vi) $3x - 4y + 1 = 0, 5x + y - 1 = 0$ and cutting off equal intercepts

from the axes.

(vii) $43x + 29y + 43 = 0, 23x + 8y + 6 = 0$ and having y-intercept -2.

(viii) $2x + 7y - 8 = 0, 3x + 2y + 5 = 0$ and making an angle of $45°$ with the line $2x + 3y - 7 = 0$.

10. Find the angles of the triangle with the given vertices $(1, 2), (3, 4)$ and $(2, 5)$.
11. What are the angles of the triangle with vertices $A(3, 2), B(4, 5)$ and $C(-1, -1)$?
12. Find the angles of triangle where the slopes of its sides, are 3, $\frac{1}{2}, -2$.

7.7 Concurrency of Straight Lines

7.7.1 Find the condition of concurrency of three straight lines

Three or more distinct lines are said to be concurrent, if they pass through the same point. The point of intersection of any two lines, which lie on the third line is called the point of concurrence.

Let the equations of the three concurrent straight lines be;

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$

$$a_3x + b_3y + c_3 = 0 \quad \ldots(iii)$$

Suppose the equations (i) and (ii) of two intersecting lines intersect at $P(x_1, y_1)$. Then (x_1, y_1) will satisfy both the equations (i) and (ii). Therefore,

$$a_1x_1 + b_1y_1 + c_1 = 0 \text{ and}$$

$$a_2x_1 + b_2y_1 + c_2 = 0$$

Solving the above two equations, we get,

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Therefore,

$$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$y_1 = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

where $a_1b_2 - a_2b_1 \neq 0$

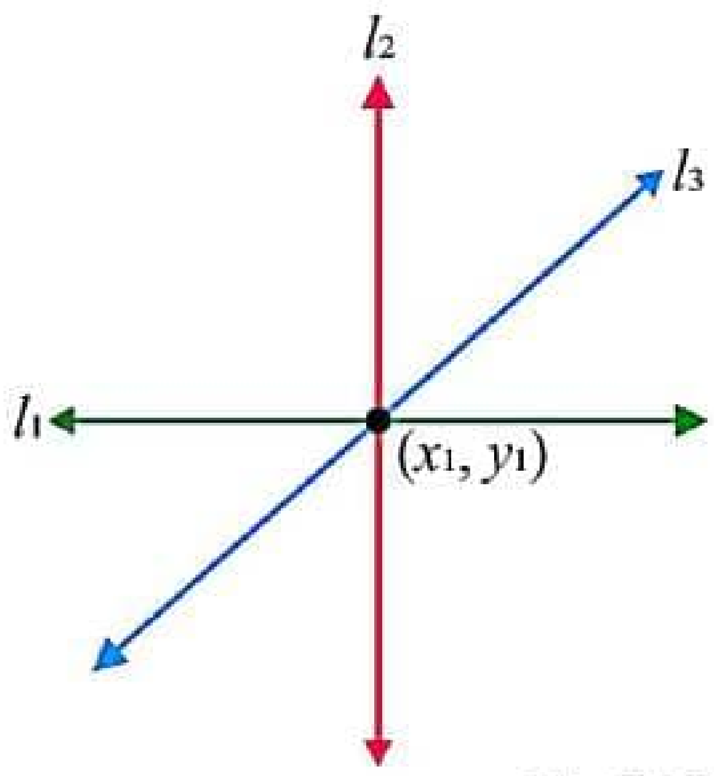


Fig 7.27

Therefore, the required co-ordinates of the point of intersection of the lines (i) and (ii) are;

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

where $a_1b_2 - a_2b_1 \neq 0$

Since the straight lines (i), (ii) and (iii) are concurrent, hence (x_1, y_1) must satisfy the equation (iii).

Therefore,

$$a_3x_1 + b_3y_1 + c_3 = 0$$

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrence of three straight lines.

The above condition is not sufficient to ensure that the three given lines are concurrent. However, it can be shown that, if the above determinant vanishes, then the lines are concurrent.

Example 1. Show that the lines $2x - 3y + 5 = 0,\ 3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$ are concurrent.

Solution: The given lines are $2x - 3y + 5 = 0,\ 3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$

We have, $$\begin{vmatrix} 2 & -3 & 5 \\ 3 & 4 & -7 \\ 9 & -5 & 8 \end{vmatrix} = 0$$

$$= 2(32 - 35) - (-3)(24 + 63) + 5(-15 - 36)$$

$$= 2(-3) + 3(87) + 5(-51)$$

$$= 0$$

Therefore, the given three straight lines are concurrent.

Example 2. For what value of 'a' the lines $2x + y - 1 = 0, ax + 2y - 2 = 0$ and $2x - 3y - 5 = 0$ are concurrent.

Solution: The given lines are $2x + y - 1 = 0, ax + 2y - 2 = 0$ and $2x - 3y - 5 = 0$,

We have;

We have, $$\begin{vmatrix} 2 & 1 & -1 \\ a & 2 & -2 \\ 2 & -3 & -5 \end{vmatrix} = 0$$

$$= 2(-10 - 6) - 1(-5a + 4) - 1(-3a - 4) = 0$$

$$= -32 + 5a - 4 + 3a + 4 = 0$$

$$a = 4$$

7.7.2 Find the equation of median, altitude and right bisector of a triangle the equations of altitudes of triangle:

Consider ΔABC be a triangle as shown in Fig. 7.28.

The equation of the altitude through vertex A (Fig. 7.28) can be calculated using the following steps;

- Find the slope of $\overline{BC}$.
- Since $\overline{AD}$ and $\overline{BC}$ are perpendicular to each other so using the coordinates of point B and C we will get;
  slope of $\overline{AD}$ × slope of $\overline{BC} = -1$.
  slope of $\overline{AD} = \dfrac{-1}{slope\ of\ \overline{BC}}$
- Using the slope intercept form of equation of a line. The equation of altitude of $\overline{AD}$ is $y = mx + c_1$, where m is the slope of $\overline{AD}$. For finding c_1 we will use the coordinates of point A.
- Thus, using the slope of equation of $\overline{AD}$, i.e. m, and y intercept c_1 we will get, the equation of altitude of $\overline{AD}$, as follows; $y = mx + c_1$

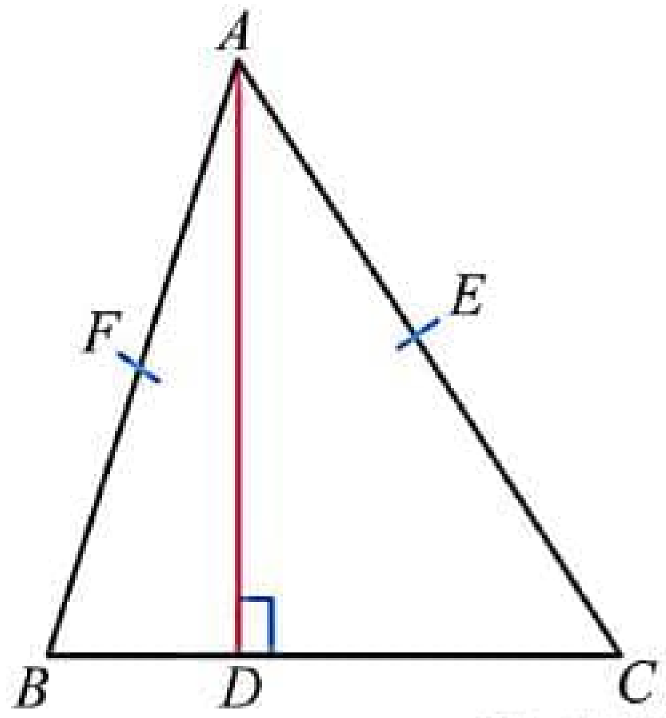


Fig 7.28

Similarly, we can find the equations of altitudes through the vertices B and C.

Example 1. $A(3,2)$ $B(6,-2)$ and $C(-7,3)$ are the vertices of ΔABC. Find the equations of the altitudes through A.

Solution: Here we have $P(x_1, y_1) = A(3,2), B(x_2, y_2) = B(6,-2)$ and $C(x_3,\ y_3) = C(-7,3)$.

For equation of the altitude through A (Fig. 7.28);

we have; slope of $\overline{BC} = \dfrac{y_2-y_1}{x_2-x_1}$

slope of $\overline{BC} = \dfrac{(3+2)}{(-7-6)}$

slope of $\overline{BC} = \dfrac{-5}{13}$

and slope of $\overline{AD}$ (where D is the point on the $\overline{BC}$) will be;

slope of $\overline{AD} = \dfrac{-1}{\frac{-5}{13}}$

slope of $\overline{AD} = \dfrac{13}{5}$

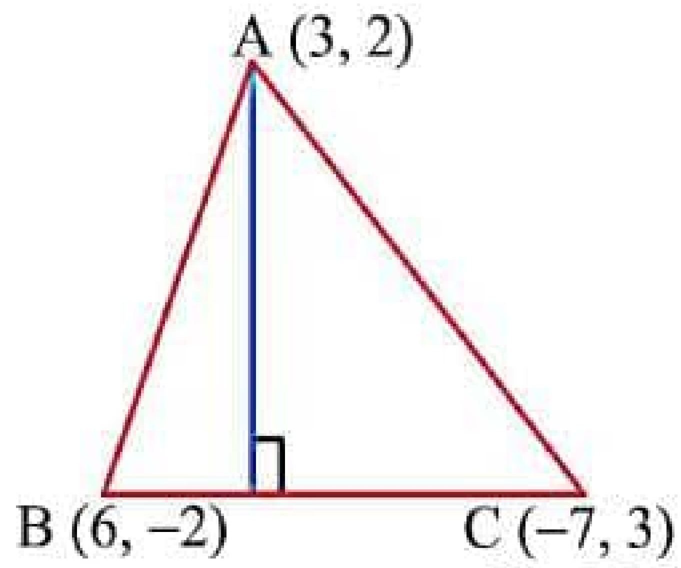


Fig 7.29

Now the equation of altitude of $\overline{AD}$

$$y = mx + c_1$$

or $$y = \frac{13}{5}x + c_1$$

For finding c_1 we will use the coordinates of point A, i.e.,

$$2 = \frac{13}{5}(3) + c_1$$

$$c_1 = \frac{39}{5} - 2 = \frac{39-10}{5} = \frac{29}{5}$$

Thus, the equation of the altitude through the point A is

$$y = \frac{13}{5}x + \frac{29}{5}$$

or $\quad 13x + 5y - 29 = 0$

- **Equation of medians of triangle**

The equation of the median through vertex A (Fig. 7.30) can be calculated using the following steps;

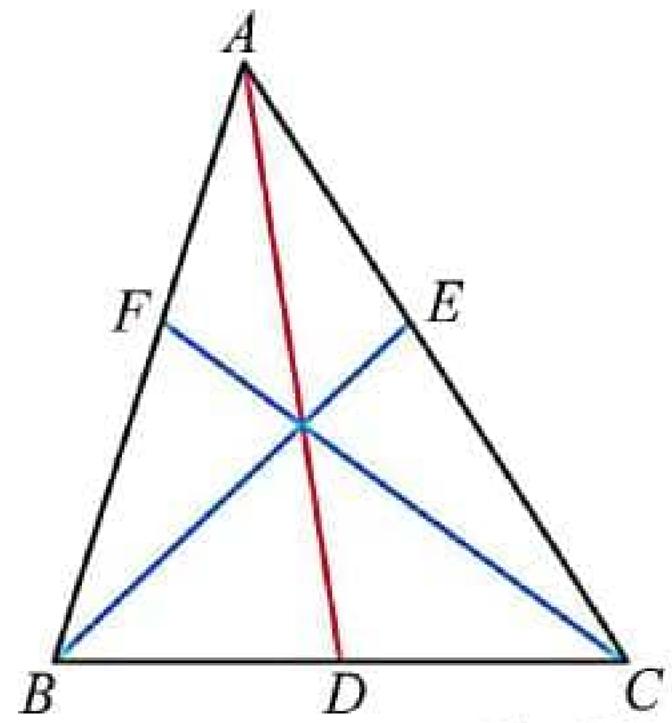


Fig 7.30

- Using midpoint formula, find the midpoint of $\overline{BC}$, which gives the coordinates of point D.
- Find the slope of median $\overline{AD}$ using the points A and D.
- Using point slope form equation $y - y_1 = m(x - x_1)$, find the equation of the median $\overline{AD}$.

Similarly, we can find the equations of medians through the vertices B and C.

Example 2. Find the equations of median of ΔABC with vertices $A(1,2)$ $B(-2,5)$ and $C(-7,4)$ through A.

Solution: Here we have $(x_1, y_1) = A(1,2), (x_2, y_2) = B(-2,5)$ and $(x_3, y_3) = C(-7,4)$.

Let D and F be the midpoints of the sides $\overline{BC}$,

Now, for the equation of Median $\overline{AD}$:

$$\text{Midpoint of } \overline{BC} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{(-2)+(-7)}{2}, \frac{5+4}{2}\right)$$

$$= \left(\frac{-9}{2}, \frac{9}{2}\right)$$

Now, find the slope of $\overline{AD}$, i.e.,

$$m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$= \frac{\frac{9}{2} - 2}{\frac{-9}{2} - 1}$$

$$= -\frac{5}{11}$$

Now, equation of median $\overline{AD}$ is as follows:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{5}{11}(x - 1)$$

or $\quad 11y - 22 = -5x + 5$

or $\quad 5x + 11y - 27 = 0$

- **Equation of the right bisector**

The equation of the right bisector through $\overline{BC}$ (Fig. 7.31) can be calculated using the following steps;

- Find the slope of $\overline{BC}$.
- Take its negative reciprocal as it is making a perpendicular line.
- Find the midpoint of $\overline{BC}$.
- Use $y = mx + c$ formula for finding the equation of line using the steps 2 and 3.

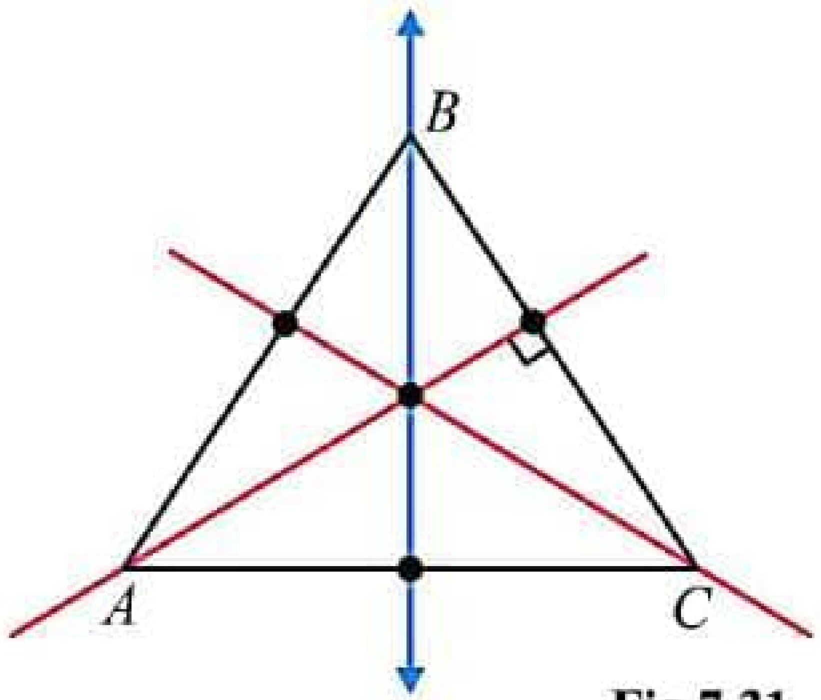


Fig 7.31

Similarly, we can find the equations of right bisectors through the vertices $\overline{AC}$ and $\overline{AB}$.

Example 3. Find the equations of right bisector of ΔABC with vertices $A(1,2)$ $B(10,-6)$ and $C(-7,2)$ through A.

Solution: Here we have $(x_1, y_1) = A(1,1), (x_2, y_2) = B(10,-6)$ and $(x_3, y_3) = C(-7,2)$.

Let D and F be the midpoints of the sides $\overline{BC}$,

Now, for the equation of right bisector through $\overline{BC}$, we have;

$$\text{slope of } \overline{BC} = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$= \left(\frac{2 - (-6)}{-7 - 10}\right)$$

$$= \left(\frac{8}{-17}\right)$$

Then its negative reciprocal is $\left(\frac{17}{8}\right)$.

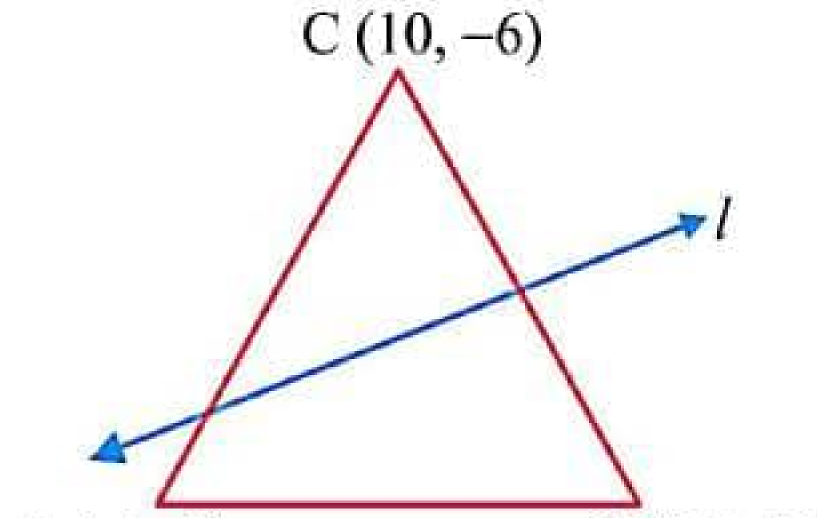


Fig 7.32

Now, find the midpoint of segment $\overline{BC}$;

$$\text{Midpoint of } \overline{BC} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{(-7) + 10}{2}, \frac{2 + (-6)}{2}\right)$$

$$= \left(\frac{3}{2}, \frac{-4}{2}\right)$$

Now, c_1 will be calculated as follows:

$$y = m(x + c_1)$$

$$-2 = \frac{17}{8}\left(\frac{3}{2}\right) + c_1$$

or $$c_1 = \frac{-83}{16}$$

$$y = \frac{17}{8}x - \frac{83}{16}$$

or $$34x - 16y - 83 = 0$$

7.7.3 Show that

- **three right bisectors,**
- **three medians,**
- **three altitudes, of a triangle are concurrent.**

- **three right bisectors of the triangle are concurrent**

Let $A(-a, 0)$, $B(a, 0)$ and $C(0, b)$ are the vertices of triangle as shown in Fig. 7.33.

Here l_1, y-axis and l_2 are the right bisectors of sides $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$ respectively. First, we find the equation of each right bisectors.

Equation of right bisector through $\overline{AB}$

Equation of right bisectors through $\overline{AB}$ is

$$x = 0 \qquad \ldots(\text{i})$$

- **Equation of right bisector through $\overline{BC}$**

Slope of $\overline{BC} = \frac{-b}{a}$

Slope of $l_1 = \frac{a}{b}$

Mid-point of $\overline{BC} = \left(\frac{a}{2}, \frac{b}{2}\right)$

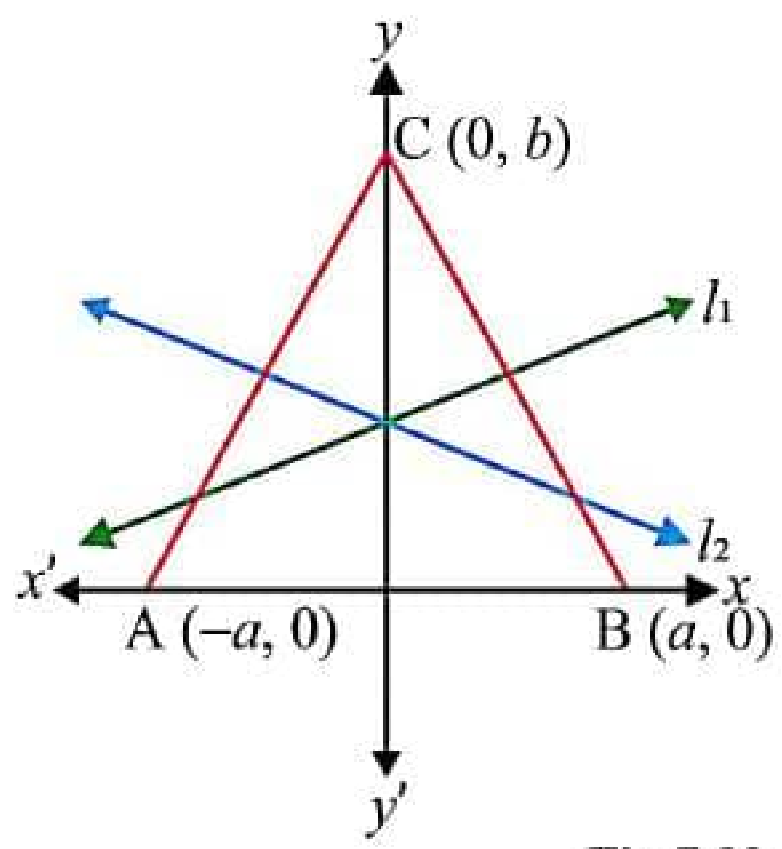


Fig 7.33

Equation of l_1 is

$$y = \left(\frac{a}{b}\right)x + c_1$$

$$\frac{b}{2} = \left(\frac{a}{b}\right)\left(\frac{a}{2}\right) + c_1$$

$$\Rightarrow \quad c_1 = \frac{b^2 - a^2}{2b}$$

Equation of right bisector through $\overline{BC}$ is

$$y = \left(\frac{a}{b}\right)x + \left(\frac{b^2 - a^2}{2b}\right) \qquad \ldots(\text{ii})$$

Similarly, equation of right bisector through $\overline{AC}$

$$y = \left(-\frac{a}{b}\right)x + \left(\frac{b^2-a^2}{2b}\right) \quad \text{...(iii)}$$

By solving equation (i) and (ii)

We get, $x = 0$ and $y = \frac{b^2-a^2}{2b}$

Similarly, by solving equation (i) and (iii)

We get, $x = 0$ and $y = \frac{b^2-a^2}{2b}$

Since $x = 0$ and $y = \frac{b^2-a^2}{2b}$ satisfy this equation (iii).

Since $\left(0, \frac{b^2-a^2}{2b}\right)$ is the intersecting point of three bisectors.

Therefore, right bisectors of triangle are concurrent.

- **Three medians of a triangle are concurrent**

Consider ΔABC be a triangle as shown in Fig. 7.34 with $A(-2a, 0), B(2a, 0)$ and $C(0, 2b)$ are its vertices.

The equations of the median through vertex A, B and C (Fig. 7.34) will be calculated as follows:

Midpoint of $\overline{AB} = F = (0, 0)$

Midpoint of $\overline{BC} = D = (a, b)$

Midpoint of $\overline{AC} = E = (-a, b)$

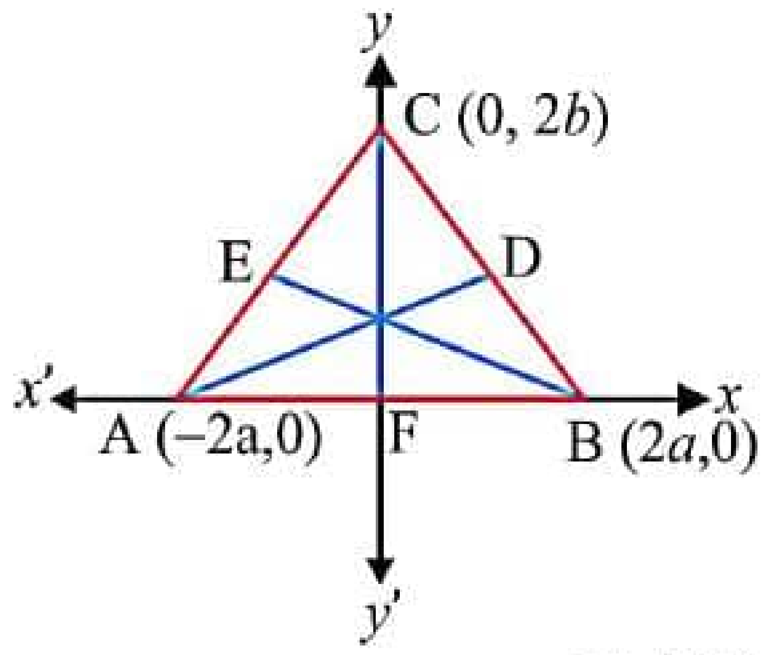


Fig 7.34

The equation of median $\overline{AD}$ by using two-point form of equation.

$$\frac{b-0}{a+2a} = \frac{y-0}{x+2a}$$

$$\Rightarrow \quad bx + 2ab = ay + 2ay$$

$$\Rightarrow \quad bx - 3ay + 2ab = 0 \quad \text{...(i)}$$

Similarly, equation of median $\overline{BE}$ is

$$bx - 3ay - 2ab = 0 \quad \text{...(ii)}$$

and equation of median $\overline{CF}$ is

$$x = 0 \quad \text{...(iii)}$$

Now the determinant of coefficient of equation (i), (ii) and (iii)

$$\begin{vmatrix} b & -3a & 2ab \\ b & -3a & -2ab \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Hence the medians of the triangle are concurrent.

- **three altitudes of a triangle are concurrent.**

Consider ΔABC be a triangle as shown in Fig. 7.35 with $A(a,0), B(b,0)$ and $C(0,c)$. Let the base $\overline{AB}$ of triangle be taken as axis of x and a line through C perpendicular to base $\overline{AB}$ is taken as the axis of y. The point of intersection is G.

The equations of the altitudes through vertex A, B and C (Fig. 7.35) will be calculated as follows:

Now, the equation of altitude through C is as follows;

$$x = 0 \quad \text{...(i)}$$

Now, we will find the slope of $\overline{BC}$ and $\overline{CA}$ of the triangle, respectively,

$$\text{Slope of } \overline{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{c-0}{0-b}$$

$$\text{Slope of } \overline{BC} = -\frac{c}{b}$$

$$\text{and Slope of } \overline{CA} = -\frac{c}{a}$$

So, the slope of the respective lines perpendicular to them are $\frac{b}{c}$ and $\frac{a}{c}$.

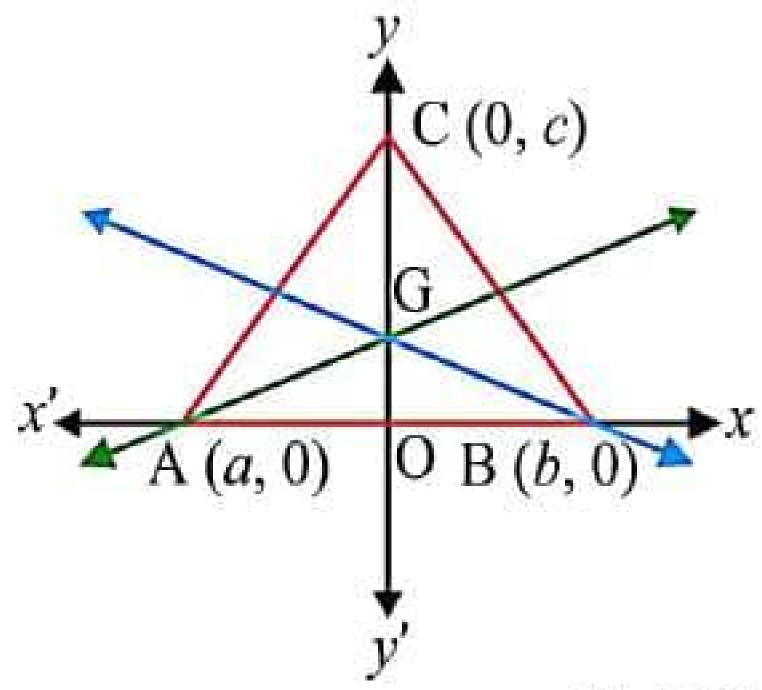


Fig 7.35

Now, the equation of altitude from A is as follows;

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{b}{c}(x - a)$$

or $$bx - cy - ab = 0 \quad \text{...(ii)}$$

Now, the equation of altitude from B is as follows;

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{a}{c}(x - b)$$

or $$ax - cy - ab = 0 \quad \text{...(iii)}$$

Now, the determinant of the coefficients of equations (i), (ii) and (iii) is

$$\begin{vmatrix} 1 & 0 & 0 \\ b & -c & -ab \\ a & -c & -ab \end{vmatrix}$$

which is zero. Hence the altitudes of a triangle are concurrent.

Exercise 7.6

1. If $A(2,5), B(3,7)$ and $C(0,8)$ are the vertices of a triangle then find the
   (i) Equation of median through A
   (ii) Equation of altitude through B
   (iii) Equation of right bisector of side $\overline{AC}$
2. Show that the following lines are concurrent. Also find their point of concurrency.
   (i) $x - y = 6, 4y + 22 = 3x$ and $6x + 5y + 8 = 0$
   (ii) $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} + \frac{y}{a} = 1$ and $y = x$.
   (iii) $5x + y + 11 = 0, x + 7y + 9 = 0$ and $2x + y + 5 = 0$.
3. If $A(-1,5), B(2,3)$ and $C(7,6)$ the vertices of triangle, then show right bisectors, medians and altitudes the triangle is concurrent.

7.8 Area of a Triangular Region

7.8.1 Find the area of a triangular region whose vertices are given

Consider ΔABC be a triangle as shown in Fig. 7.36 with the coordinates of triangle in anticlockwise direction, i.e.; $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$.

Let $\overline{AD}$ be perpendicular to $\overline{BC}$ (Fig. 7.36). By using the elementary geometry, the area of triangle ABC, i.e.; ▲;

$$\blacktriangle = \frac{1}{2} \times (\text{base}) \times (\text{Altitude})$$

$$= \frac{1}{2} \times |\overline{BC}| \times |\overline{AD}|$$

where $\overline{BC} = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$

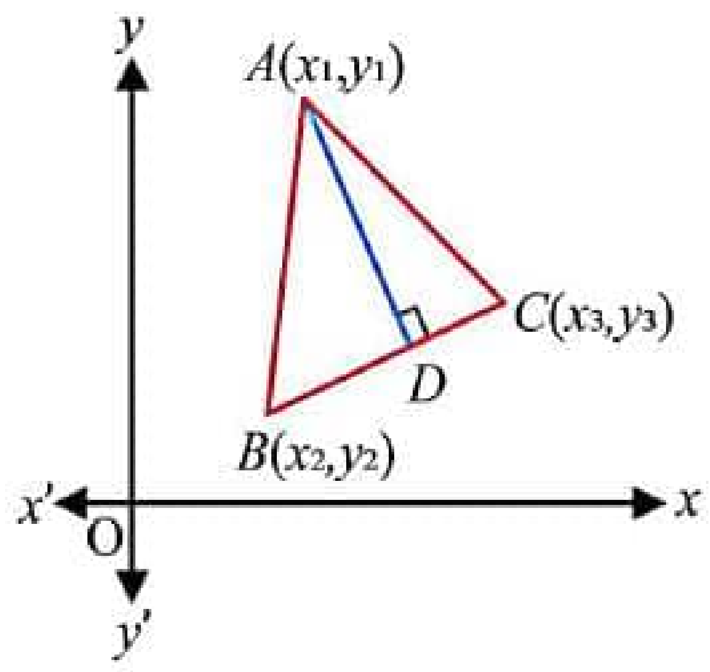


Fig 7.36

Also, the equation of the line l of which $\overline{BC}$ is a segment can be found, as it passes through $B(x_2, y_2)$ and $C(x_3, y_3)$.

So, the equation of l is;

$$y - y_2 = \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

or $\quad (y_2 - y_3)x + (x_3 - x_2)y + x_2y_3 - x_3y_2 = 0$

Now $|\overline{AD}|$ = The perpendicular distance of $A(x_1, y_1)$ from the line l is;

$$= \frac{((y_2 - y_3)\,x_1 + (x_3 - x_2)y_1 + x_2y_3 - x_3y_2}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}$$

Thus, area of triangle ABC is;

$$\blacktriangle = \frac{1}{2} \times (\text{base}) \times (\text{Altitude})$$

$$= \frac{1}{2}\{(y_2 - y_3)\,x_1 + (x_3 - x_2)y_1 + (x_2y_3 - x_3y_2)\}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Using the properties of determinant this result can also be written as;

$$\Delta = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

Corollary 1: If A, B and C are three collinear points, then the area of triangle ABC = 0, i.e., the condition for collinearity of three points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ is;

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Corollary 2: The area of polygon whose vertices taken in order in anticlockwise direction are; (x_1, y_1), (x_2, y_2), (x_3, y_3) ... (x_n, y_n)

$$= \frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\}$$

Area of a quadrilateral can thus be written as

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$$

Example: Find the area of the triangle whose vertices are

(i) $(2, 9), (-2, 1)$ and $(6, 3)$ (ii) $(3, 8), (7, 2)$ and $(-1, 1)$

Solution (i): The area of triangle with the vertices $(2, 9), (-2, 1)$ and $(6, 3)$ is given by;

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 9 & 1 \\ -2 & 1 & 1 \\ 6 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{2(1 - 3) - 9(-2 - 6) + 1(-6 - 6)\}$$

$$= \frac{1}{2}(56)$$

$= 28$ square units.

Solution (ii): The area of triangle with the vertices $(3, 8), (7, 2)$ and $(-1, 1)$ is given by;

$$\blacktriangle = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ 7 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{3(2 - 1) - 8(7 + 1) + 1(7 + 2)\}$$

$= \frac{1}{2}(-52)$

$= 26$ square units.

Thus, omitting the negative sign, the magnitude of the area $= 26$ square units.

Exercise 7.7

1. Find the area of the triangle whose vertices are:
   (i) $(11,-12),(6,2)$ and $(-5,10)$ (ii) $(3,1),(-2,5)$ and $(-4,-5)$
   (iii) $(-5,-2),(4,-6)$ and $(1,7)$
   (iv) $(-a,b+c),(a,b-c)$ and $(a,-c)$
   (v) $(a\cos\theta_1, b\sin\theta_1),(a\cos\theta_2, b\sin\theta_2)$ and $(a\cos\theta_3, b\sin\theta_3)$
2. Find the area of a quadrilateral whose consecutive vertices are;
   (i) $(3,-3),(7,5),(1,2)$ and $(-3,4)$
   (ii) $(2,3),(-1,2),(-3,2)$ and $(3,-3)$
3. Prove, by the method of the area of a triangle, that the following points are collinear;
   (i) $(2,3),(5,0)$, and $(4,1)$ (ii) $(2,1),(4,-1)$, and $(1,2)$
   (iii) $(-1,-1),(5,7)$ and $(8,11)$
4. Find the area of triangle formed by the lines;
   (i) $y=0, y=2x$ and $y=6x+5$
   (ii) $y-x=0, y+x=0$ and $x-c=0$.
   (iii) $y=2x+3, 2y+3x=3$ and $x+y+2=0$.

7.9 Homogenous Equation

7.9.1 Recognize homogeneous linear and quadratic equations in two variables

When a straight line passes through the origin, then its equation will become $ax+by=0$ and is known as homogeneous linear equation in two variables.

Similarly, in general quadratic equation in two variables

$ax^2+by^2+2hxy+2gx+2fy+c=0$

if $g=f=c=0$ then it becomes homogeneous quadratic equation in two variables, we write as $ax^2+2hxy+by^2=0$

7.9.2 Investigate that the 2nd degree homogeneous equation in two variables x and y represents a pair of straight lines through the origin and find acute angle between them.

Let l_1 and l_2 are two straight lines passing through the origin and $y=m_1x$ and $y=m_2x$ are the equations of l_1 and l_2 respectively, as shown in the figure 7.37.

Now, take any point P on the line l_1. The path it travels on these two lines is its locus. To find the equation of the locus we multiply equation of l_1 and l_2

i.e., $$(y - m_1x)(y - m_2x) = 0$$

$$\Rightarrow \quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$$

is the equation pair of straight lines passing through the origin, which is homogeneous second order quadratic equation.

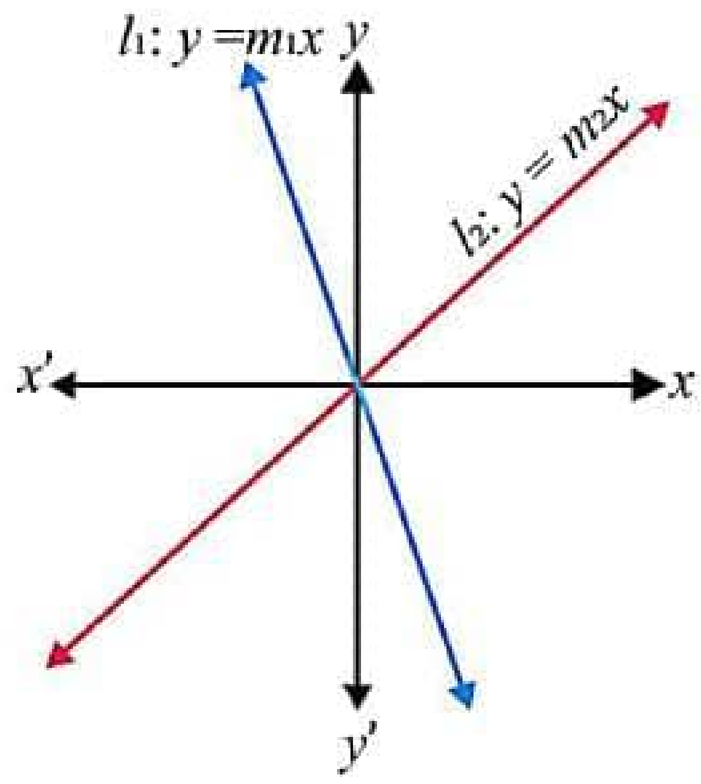


Fig. 7.37

Theorem 1: A homogeneous equation of degree two in x and y, i.e., $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin if $h^2 - ab \geq 0$.

Proof: Let the second-degree homogeneous equation in x and y be $ax^2 + 2hxy + by^2 = 0$ where a, h, b are real numbers and not all zero.

Case I: Let $a = 0$ and $b = 0$, but $h \neq 0$.

$hxy = 0$ as $h \neq 0, xy = 0$

$xy = 0 \Rightarrow x = 0$ or $y = 0$

Separate equations are $x = 0$ and $y = 0$ which are the equations of the coordinate axes.

$ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through origin when $a = 0$ and $b = 0$.

Case II: Let $a = 0$, Given equation becomes $2hxy + by^2 = 0$

$y(2hx + by) = 0$ gives us $y = 0$ and $2hx + by = 0$.

The two factors are linear in x and y and do not contain constant terms.

Hence $ax^2 + 2hxy + by^2 = 0$ represent a pair of lines through origin for $a = 0$. The same thing can be proved by taking $b = 0$.

Case III: Let $a \neq 0$ Now, $ax^2 + 2hxy + by^2 = 0$, multiply both sides of equation by a;

$$a^2x^2 + 2ahxy + aby^2 = 0$$

$$a^2x^2 + 2(ax)(hy) + aby^2 = 0$$

$$\Rightarrow \quad a^2x^2 + 2(ax)(hy) + h^2y^2 + (aby^2 - h^2y^2) = 0$$

$$\Rightarrow \quad (ax + hy)^2 - y^2(h^2 - ab) = 0$$

$$\Rightarrow \quad (ax + hy)^2 - \left(y\sqrt{(h^2 - ab)}\right)^2 = 0$$

$$\left(ax + hy - y\sqrt{(h^2 - ab)}\right)\left(ax + hy + y\sqrt{(h^2 - ab)}\right) = 0$$

The two factors are linear in x and do not contain constant term. and lines will be real if and only if $h^2 - ab \geq 0$. It is a pair of lines. Their separate equations

are $(ax + hy - y\sqrt{(h - ab)} = 0$ and $\left(ax + hy + y\sqrt{(h^2 - ab)}\right) = 0$

which are separately satisfied by the origin. Hence $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through origin, for $a \neq 0$.

Where their slopes are

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b}$$

$$m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

Combining the cases (I), (II) and (III) we get that every second-degree homogeneous equation in x and y in general represents a pair of lines through origin.

Theorem 2: If m_1 and m_2 are the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$ show that $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

Deduce that lines are perpendicular if $a + b = 0$ and lines are coincident if $h^2 = ab$.

Proof: m_1 and m_2 are the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$, Equation of first line is $y = m_1 x$, i.e., $m_1 x - y = 0$ and equation of second line is $y = m_2 x$, i.e., $m_2 x - y = 0$. Combined equation is:

$$(m_1 x - y)(m_2 x - y) = 0$$

$$m_1 x(m_2 x - y) - y(m_2 x - y) = 0$$

or $$m_1 m_2 x^2 - (m_1 + m_2)y - y^2 = 0$$

Also $ax^2 + 2hxy + by^2 = 0$ is combined equation of the two lines. The equation $ax^2 + 2hxy + by^2 = 0$ and $m_1 m_2 x^2 - (m_1 + m_2)y - y^2 = 0$ are identical.

Hence their corresponding coefficients are proportional.

$$\frac{a}{m_1 m_2} = \frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

Now

$$\frac{a}{m_1 m_2} = \frac{b}{1}$$

$$m_1 m_2 = \frac{a}{b}$$

Now

$$\frac{2h}{-(m_1 + m_2)} = \frac{b}{1}$$

$$m_1 + m_2 = \frac{-2h}{b}$$

Case – I: Lines are perpendicular if $m_1 m_2 = -1$

$$\frac{a}{b} = -1$$

$$a = -b$$
$$a + b = 0$$

Case – II: Lines are coincident if $m_1 = m_2$

$$m_1 - m_2 = 0$$
$$(m_1 - m_2)^2 = 0$$

If $\qquad (m_1 + m_2)^2 - 4m_1m_2 = 0$

Then substituting the values we will get;

$$\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b} = 0$$
$$\frac{4h^2}{b^2} - 4\frac{a}{b} = 0$$
$$h^2 - ab = 0 \quad \Rightarrow \quad h^2 = ab$$

Nature of Roots:

- If $h^2 > ab$ then the roots of the equation are real and distinct.
- If $h^2 = ab$ then the roots of the equation are real and equal.
- If $h^2 < ab$ then the roots of the equation are imaginary.

Pair of Straight Lines Formulas:

We know that angle between two lines is:

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1m_2}$$

and on substituting the value of $m_1 + m_2 = \frac{-2h}{b}$ and $m_1m_2 = \frac{a}{b}$,

we get,

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{b}\right)$$

which is the angle between the two lines expressed as $ax^2 + 2hxy + by^2 = 0$.

Example: Find the equations of the pair of lines represented jointly by the given equation. State nature of lines and also find the acute angle between lines $5x^2 + 13xy - 6y^2 = 0$.

Solution: Factorizing the given equation, we get;

$$5x^2 + 13xy - 6y^2 = 0 \Rightarrow (x + 3y)(5x - 2y) = 0$$

So, the given equations represent the lines $x + 3y = 0$ and $5x - 2y = 0$.

Now compare the given equation with $ax^2 + 2hxy + by^2 = 0$, so, $a = 5, 2h = 13$ and $b = -6$.

Put the values in $D = h^2 - ab \Rightarrow D = \frac{73}{2}$.

Hence $D > 0$ then the lines are real and distinct.

The acute angle between the line

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{b}$$

$$\tan\theta = \frac{2\sqrt{\left(\frac{13}{2}\right)^2 - 5(-6)}}{(-6)} = -\frac{17}{6}$$

$$\theta = \tan^{-1}\left(\frac{17}{6}\right) = 70.55°$$

Exercise 7.8

1. Find the equations of the pair of lines represented jointly by each of the following equations. State nature of lines and also trace the pair of lines;

   (i) $x^2 - 5xy + 6y^2 = 0$ (ii) $4x^2 - xy - 5y^2 = 0$

   (ii) $9x^2 - 6xy + y^2 = 0$ (iii) $10x^2 - 3xy - y^2 = 0$

   (v) $7x^2 - 3xy + 5y^2 = 0$

2. Find the combined equation of the pair of lines through the origin which are perpendicular to the lines represented by

   (i) $2x^2 - 5xy + y^2 = 0$ (ii) $6x^2 - 13xy + 6y^2 = 0$

3. Trace the pair of lines given by the following equations;

   (i) $x^2 - 3xy + 2y^2 = 0$ (ii) $x^2 - 6xy + 9y^2 = 0$

   (iii) $6x^2 - xy - y^2 = 0$ (vi) $8x^2 - 3xy - y^2 = 0$

4. Find the angle between the lines represented by;

   (i) $x^2 - 5xy + 6y^2 = 0$ (ii) $3x^2 + 7xy + 2y^2 = 0$

   (iii) $x^2 + 2xy - 3y^2 = 0$ (iv) $x^2 + xy - 2y^2 = 0$

5. The gradient of one of the lines of $ax^2 + 2hxy + by^2 = 0$ is twice that of the other. Show that $8h^2 = 9ab$.

Review Exercise 7

1. Select correct option.

   (i) The slope of the line that passes through $(-3,-4)$ and $(7,6)$ is

   (a) 0 (b) undefined (c) -1 (d) 1

   (ii) A line with a slope of $\frac{1}{2}$ and a y-intercept 7 is

   (a) $2y = x - 7$ (b) $y = \frac{1}{2}x + 7$

   (c) $y = \frac{1}{2}x - 7$ (d) $x - 2y = 14$

(iii) Two lines are said to be parallel if and only if their slopes are

(a) Equal (b) Unequal

(c) Does not exit (d) negative reciprocals of each other

(iv) Two lines l_1 and l_2 are said to be perpendicular if and only if

(a) $m_1m_2 = -1$ (b) $m_1m_2 = 1$ (c) $m_1 = -m_2$ (d) $m_1 = \frac{1}{m_2}$

(v) The slope of a line that is perpendicular to a vertical line is

(a) 0 (b) 1 (c) 90° (d) undefined

(vi) The slope of the line which makes an angle 45° with the line $3x - y = -5$ is

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{2}, -2$ (d) -2

(vii) The point on the line $2x - 3y = 5$ is equidistant from $(1, 2)$ and $(3, 4)$ is

(a) $(-2, 2)$ (b) $(4, 1)$ (c) $(1, -1)$ (d) $(4, 6)$

(viii) In a plane three or more points are said to be collinear if

(a) they lie on a circle (b) they form a closed loop together

(c) they lie on a straight line (d) they do not make any defined shape

(ix) If the line coincides with x-axis then its equation is

(a) $y = b$ (b) $-b$ (c) $y = 0$ (d) ∞

(x) The general equation of line also known as standard equation of line is,

(a) $ax + by + c = 0$ (b) $y = ax + c$

(c) $y - y_1 = m(x - x_1)$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

2. If the distance between the points $(5, -2)$ and $(1, a)$ is 5, find the values of a.
3. $M(3, 8)$ is the midpoint of the line AB. A has the coordinates $(-2, 3)$, find the coordinates of B.
4. The diameter of a circle has endpoints: $(2, -3)$ and $(-6, 5)$. Find the coordinates of the center of this circle?
5. Find the coordinates of the points which divides the join of $P(-1, 7)$ and $Q(4, -3)$ in the ratio $2 : 3$.
6. Find the points of trisection of the line segment AB, where $A(-6, 11)$ and $B(10, -3)$.
7. Two vertices of a triangle are $(1, 4)$ and $(3, 1)$. If the centroid of the triangle is the origin, find the third vertex.
8. Find the slope of the line which is perpendicular to the given line whose equation is $-2y = -8x + 9$.
9. If a straight line intercepts the x-axis at (6,0) and intercepts the y-axis at $(0, 5)$, write the equation ofthe straight line in two intercept form.

10. Determine the slope (gradient) and y-intercept of each line.

(a) $y = 12x - 6$ (b) $y = 5 - 2x$

(c) $4x - y + 13 = 0$ (d) $y = 4x$

11. Find the x and y-intercept of the equation $4x + 8y + 2 = 0$.

12. What is the value of k so that the line through $(3, k)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?

13. Tell whether the following pair of lines is parallel, perpendicular or neither?

(a) $y = 2x + 7$ and $y = 7x - 2$ (b) $x = 1$ and $x = 5$.

(c) $x = -3$, and $y = -3$

14. For what value of k will the three lines $y = 3x - 1$, $2y = x + 3$ and $3y = kx + 4$ be concurrent.

15. Find the value of a and b for which the lines given by $ax - 2y - 1 = 0$ and $6x - 4y - b = 0$

(i) are parallel (ii) are perpendicular

(iii) coincided (iv) have no common point.

16. Find the equation of the line passing through the intersection of the lines $x + 2y - 5 = 0$ and $3x - 2y + 1 = 0$ and

(i) parallel to the line $4x + 3y - 5 = 0$

(ii) perpendicular to the line $2x - 3y + 7 = 0$.

17. Find the distance of the point from the line

(i) $15x - 8y - 5 = 0$, $(2, 1)$ (ii) $2x - 7y + 1 = 0, (7, 4)$

Unit 8 Circle

8.1 Conics

Introduction to conics

According to the Greek mathematicians, conics or conic sections are the curves that can be obtained as intersections of a cone and a plane, the most important of which are circles, ellipses, parabolas and hyperbolas.

With the advent of analytic geometry and calculus, conics got great importance in the physical sciences. In 1609 Johannes Kepler presented his landmark discovery that the path of each planet about the sun is an ellipse. Galileo and Newton showed that objects under the gravitational forces can also move along paths that are parabolas and hyperbolas.

Nowadays, properties of conics are used in the construction of telescopes, radar antennas, medical equipment, navigational systems and in the determination of satellite orbits.

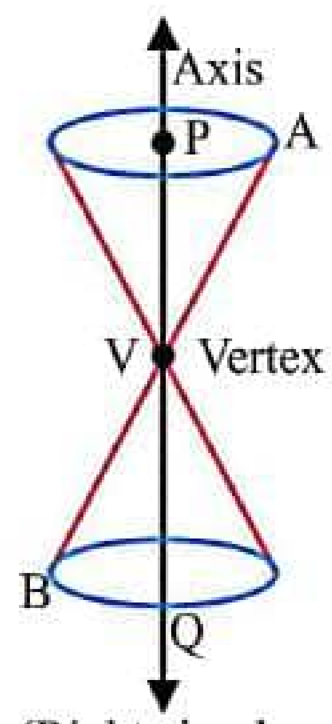
8.1.1 Define conics and demonstrate members of its family, i.e., circle, parabola, ellipse and hyperbola

As discussed earlier, the Greek mathematicians studied conics and defined them as sections of a right circular cone by planes.

In analytic geometry, a conic is the path or locus of a point moving so that the ratio of its distance from a fixed point to the distance from a fixed line is constant.

A double right circular cone or simply a cone is the surface in three-dimensional space which is generated by all the lines through a fixed point, called "vertex" and the circumference of a circle as shown in Fig. 8.1.

All such lines are called generators. The line through the centre of the circle and perpendicular to its plane is called "axis" of the cone. It also passes through vertex of the cone.



(Right circular cone)
Fig. 8.1

In Fig. 8.1, a double right circular cone or simply a cone is shown which has two parts called nappes. $\overleftrightarrow{PQ}$ is the axis and $\overline{AB}$ is a generator of the cone, whereas V is the vertex.

A circle is a conic or curve which is obtained by cutting a cone with a plane that is perpendicular to the axis and does not contain the vertex as shown in Fig. 8.2.

An ellipse is a conic or curve which is obtained if the intersecting plane is slightly tilted and cuts only one nappe of the cone as shown in Fig. 8.3.

A parabola is a conic or curve which is obtained if the cutting plane is parallel to a generator of the cone but intersects its one nappe only as shown in Fig. 8.4.

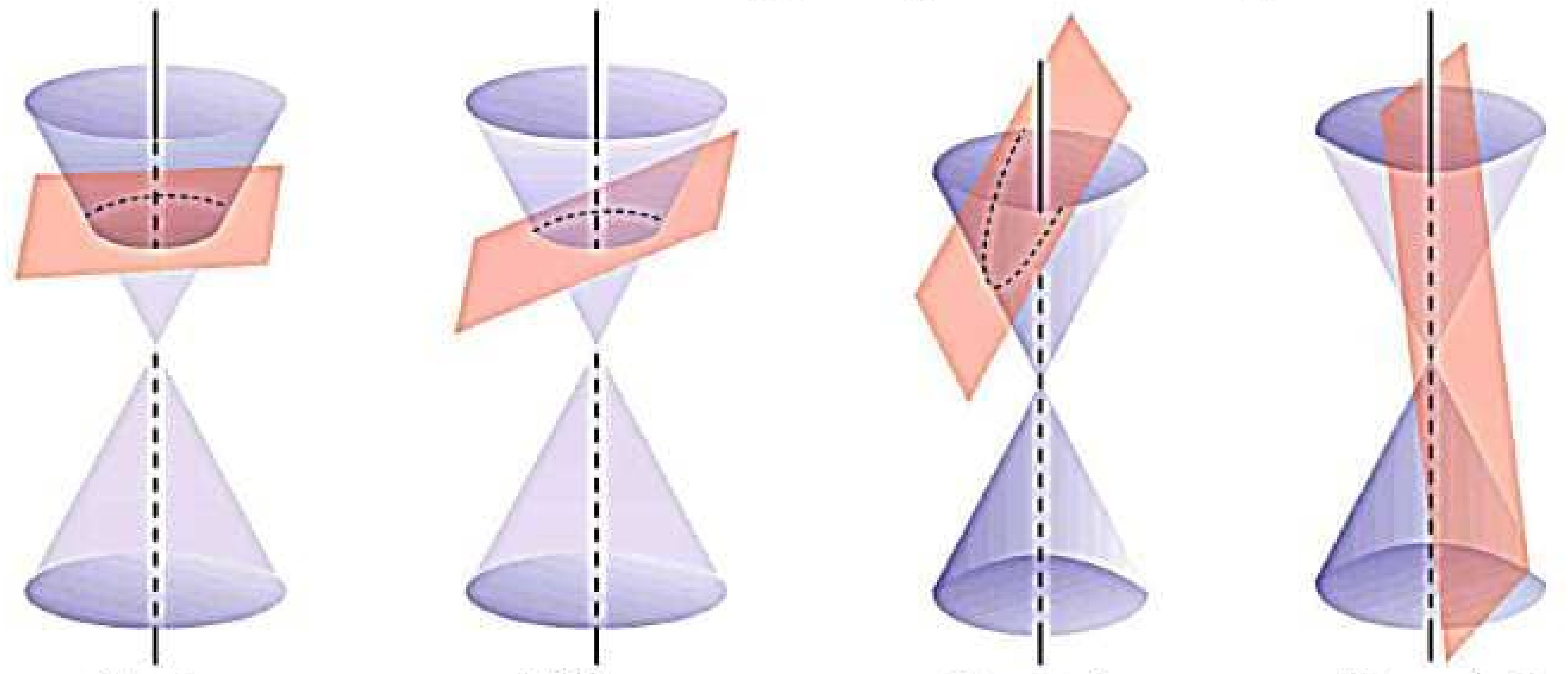


Fig. 8.2 **Fig. 8.3** **Fig. 8.4** **Fig. 8.5**

If the plane intersects both nappes but does not contain the vertex, the resulting intersection is a hyperbola as shown in Fig. 8.5.

If the cutting plane passes through the vertex then it is possible to obtain a point, a line or a pair of lines. These are called degenerate conics as shown in Fig. 8.6.

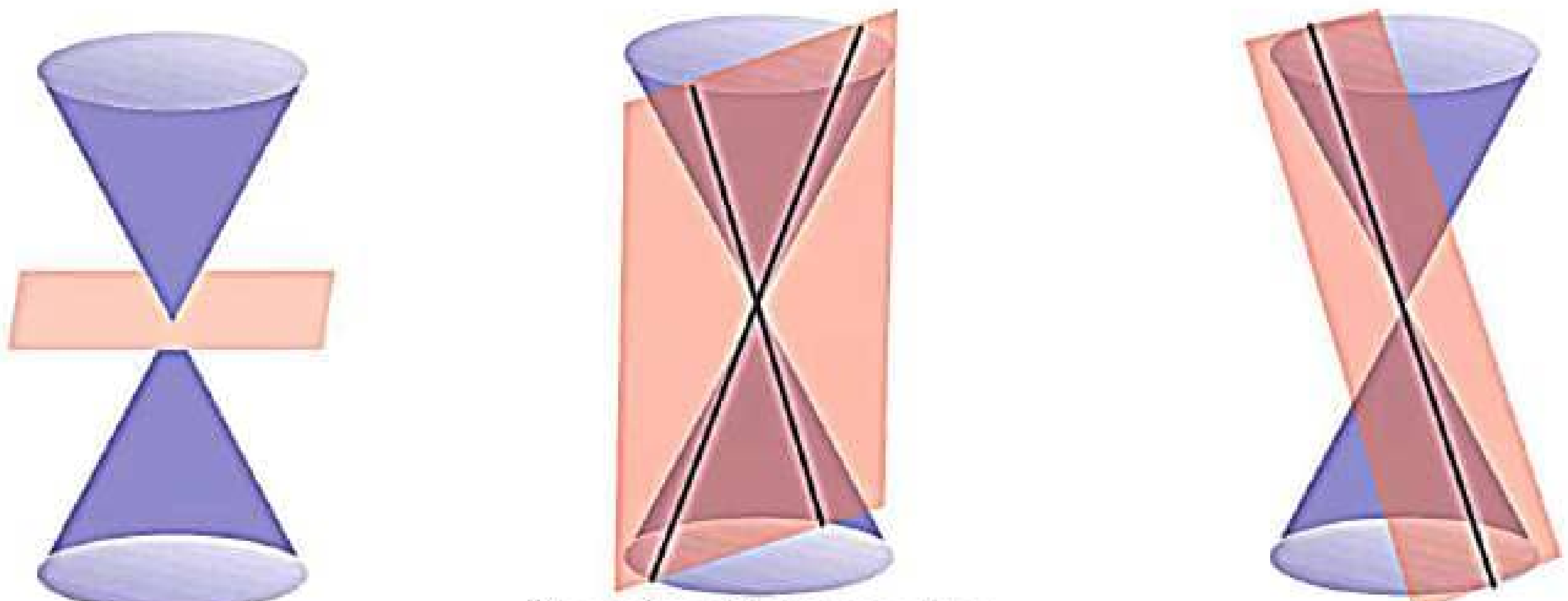


Fig. 8.6

8.2 Circle and its standard form of Equation

We know that curves in plane are described by their equations. Likewise, circle has its own equation which has different forms. First, we will discuss standard form of the equation of circle.

8.2.1 Define circle and derive its equation in standard form

i.e., $(x-h)^2+(y-k)^2=r^2$

We are already familiar with the concept of circle and its related terms. Let us recall the definition of circle.

A circle is a set of the points in plane which are equidistant from a given fixed point.

The fixed point is called the centre of the circle and the constant distance of each point of circle from the centre is called radius of the circle.

- **Standard form of equation of circle:**

Let $C(h,k)$ be the centre and r the radius of a circle as shown in the figure 8.7.

If $P(x,y)$ is any point on the circle then using distance formula.

We have $\overline{|CP|} = \sqrt{(x-h)^2 + (y-k)^2}$

or $r = \sqrt{(x-h)^2+(y-k)^2}$ $\quad (\because \overline{|CP|} = r)$

squaring both sides

we get $(x-h)^2 + (y-k)^2 = r^2$ ...(i)

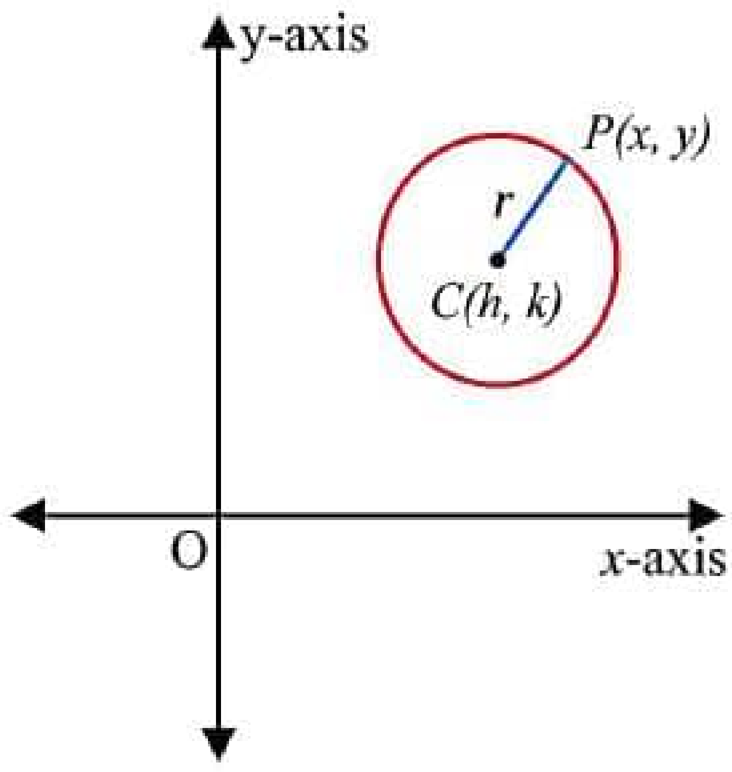


Fig. 8.7

Equation (i) represents the circle with centre (h,k) and radius r. This equation is called standard form of equation of circle or standard equation of circle.

If $(h,k) = 0$ then equation (i) becomes

$$x^2 + y^2 = r^2 \qquad \text{...(ii)}$$

Equation (ii) represents the circle with centre at origin and radius r

Example 1. Find equation of the circle whose centre is at $(-3,5)$ and radius $\sqrt{2}$ units.

Solution:

Here $(h,k) = (-3,5)$

and $r = \sqrt{2}$ units

Using standard form of equation of circle.

We get $(x+3)^2 + (y-5)^2 = \left(\sqrt{2}\right)^2$

$\Rightarrow x^2 + 6x + 9 + y^2 - 10y + 25 = 2$

$\Rightarrow x^2 + y^2 + 6x - 10y + 32 = 0$

Example 2. $P(3,4)$ is the point of a circle with centre at origin. Find the radius of the circle.

Solution: Let r be the radius of the circle. We know that equation of circle with centre at origin and radius r is

$$x^2 + y^2 = r^2 \qquad \text{... (i)}$$

$\because P(3,4)$ lies on this circle

$\therefore$ equation (i) becomes

$3^2 + 4^2 = r^2$

$\Rightarrow r^2 = 25$

$\Rightarrow r = 5$ units.

So, the radius of the circle is 5 units.

8.3 General Form of an Equation of a Circle

General form of an equation of a circle is infact the simplified form of standard equation of circle. Here, we will recognize its equation and find its centre and radius.

8.3.1 Recognize general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and find its centre and radius

Let us consider the standard equation of circle with centre (h, k) and radius r

$$(x-h)^2 + (y-k)^2 = r^2$$

On simplification, we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

using $\quad h = -g, k = -f$ and $h^2 + k^2 - r^2 = c$

we get $\quad \boxed{x^2 + y^2 + 2gx + 2fy + c = 0} \quad$ ...(i)

This equation is called the general form of the equation of circle or simply general equation where

$g, f, c \in \mathbb{R}$ and $g^2 + f^2 - c \geq 0$

In order to find centre and radius of the circle of equation (i)

We convert equation (i) in standard form.

From equation (i), we get

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) - g^2 - f^2 + c = 0$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = g^2 + f^2 - c \quad \text{... (ii)}$$

Comparing equation (ii) with standard equation of circle

We get centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

If we compare general equation of second degree i.e.,

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \quad$ ...(iii)

with general equation of circle represented by equation (i), we notice that equation (iii) represents a circle if $h = 0$ and $a = b = 1$.

In case $a = b = k$ and $h = 0$, equation (iii) becomes

$kx^2 + ky^2 + 2gx + 2fy + c = 0 \quad$ ...(iv)

which is also equation of circle, equation (iv) will take form of equation (i) if we divide both sides by k.

Example 1. Find centre and radius of the circle $x^2 + y^2 + 14x - 6y - 6 = 0$.

Solution: Comparing given equation of circle with general equation, we get

$\Rightarrow \quad 2g = 14, 2f = -6$ and $c = -6$

i.e., $\quad g = 7,\ f = -3$ and $c = -6$

We know that

$$\text{Centre} = (-g, -f) = (-7, 3)$$

and

$$\text{radius} = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{49 + 9 + 6}$$
$$= \sqrt{64} = 8 \text{ units.}$$

Thus, centre of circle is $(-7, 3)$ and radius 8 units.

Example 2. Find the value of k, if radius of the circle $3x^2 + 3y^2 - 18x + 12y + k = 0$ is 5 units.

Solution: We have

$$3x^2 + 3y^2 - 18x + 12y + k = 0$$

Dividing both sides, by 3

so, we get

$$x^2 + y^2 - 6x + 4y + \frac{k}{3} = 0$$

Comparing with the general equation of circle

$$2g = -6, 2f = 4 \text{ and } c = \frac{k}{3}$$

we get $\quad g = -3, f = 2$ and $c = \frac{k}{3}$

We know that

$$r = \sqrt{g^2 + f^2 - c}$$

so, $\quad 5 = \sqrt{9 + 4 - \frac{k}{3}} \qquad (\because r = 5 \text{ units})$

$\Rightarrow \quad 25 = 13 - \frac{k}{3}$

$\Rightarrow \quad -12 = \frac{k}{3}$

or $\quad k = -36$

so, the value of k is -36.

Exercise 8.1

1. Describe the condition under which a plane cuts right circular cone to produce.
   (i) circle (ii) parabola (iii) ellipse
   (iv) hyperbola (v) a degenerate conic
2. Find the equation of the circle if:
   (i) Centre is at origin and radius $5\sqrt{2}$ units.
   (ii) Centre is $(-5, 7)$ and radius 6 units.

(iii) $(2,-3)$ and $(-4,7)$ are the ends of its diameter.

(iv) Centre is at origin and contains a point $(5,6)$.

(v) Centre is at $(2,3)$ and contains the point $(5,7)$.

(vi) Centre is at (p,q) and radius of $\sqrt{p^2+q^2}$ units.

3. Find the centre and radius of each of the following circles. Also draw the circles.

(i) $x^2+y^2-25=0$

(ii) $(x+3)^2+(y-5)^2=49$

(iii) $x^2+y^2-6x+8y+10=0$

(iv) $x^2+y^2-8x+9=0$

(v) $5x^2+5y^2+20x-15y+10=0$

4. Find the value of k if the radius of the following circle is 10 units.

$2x^2+2y^2-8x+4y+3k=0$

5. Find the equation of the circle passing through $(-3,-4)$ and is concentric with the circle whose equation is $x^2+y^2-6x+8y-24=0$. Also identify the outer circle.

6. Show that the equation $x=a\cos\theta$ and $y=a\sin\theta$ represent a circle with centre at origin and radius equal to a

7. Prove that the equation of a circle

(i) through the origin has no constant term.

(ii) with centre on x-axis has no term in y.

(iii) with centre on y-axis has no term in x.

(iv) with centre at origin has no term in x and y both.

8.4 Equation of Circle determined by a given condition

In this section, we will find the equation of circle which is determined by the following conditions.

8.4.1 Find the equation of a circle passing through:

- **three non-collinear points,**
- **two points and having its centre on a given line,**
- **two points and equation of tangent at one of these points is known,**
- **two points and touching a given line.**

(a) Equation of circle passing through three non-collinear points

We know that one and only one circle can pass through three non-collinear points. So a unique circle can be determined if three non-collinear points are given. The method of finding equation of circle under this condition is explained in the following example.

Example: Find the equation of circle through the points $(3,0),(0,-2),(-3,4)$.

Solution: Let the equation of given circle be

$x^2+y^2+2gx+2fy+c=0$ ...(i)

$\because$ $(3,0)$ lines on it

$\therefore$ $9+6g+c=0$ ... (ii)

As $(0,-2)$ and $(-3,4)$ also lie on it. So, we have

$4-4f+c=0$ ... (iii)

$25-6g+8f+c=0$ ... (iv)

Subtracting equation (iii) from equation (ii)

we get $5+6g+4f=0$ ...(v)

Subtracting equation (iii) from equation (iv)

we get $21-6g+12f=0$ ...(vi)

Solving equation (v) and (vi) simultaneously,

we get $f=-\frac{13}{8}$ and $g=\frac{1}{4}$

From equation (iii), we get $c=-\frac{21}{2}$

By using these values of g,f and c in equation (i)

We get $x^2+y^2+\frac{x}{2}-\frac{13y}{4}-\frac{21}{2}=0$

or $4x^2+4y^2+2x-13y-42=0$

This is the required equation of the circle.

(b) Equation of a circle passing through two points and having its centre on a given line.

We know that infinitely many circles can be drawn from two points but particular circle or circles can be obtained under certain condition. In the following example we find the equation of a circle passing through two points with the condition that its centre lies on a given line.

Example: Find the equation of a circle passing through two points $(1,4)$ and $(3,2)$ and having its centre on the line $2x+y-1=0$.

Solution: Let equation of the circle be

$x^2+y^2+2gx+2fy+c=0$ ...(i)

The circle passes through the points $(1,4)$ and $(3,2)$.

$\therefore$ we have

$17+2g+8f+c=0$ ...(ii)

and $13+6g+4f+c=0$ ...(iii)

As centre $(-g,-f)$ of the circle lies on the line

$2x + y - 1 = 0$

So, we have $\quad -2g - f - 1 = 0$

or $\quad 2g + f + 1 = 0 \quad \ldots(\text{iv})$

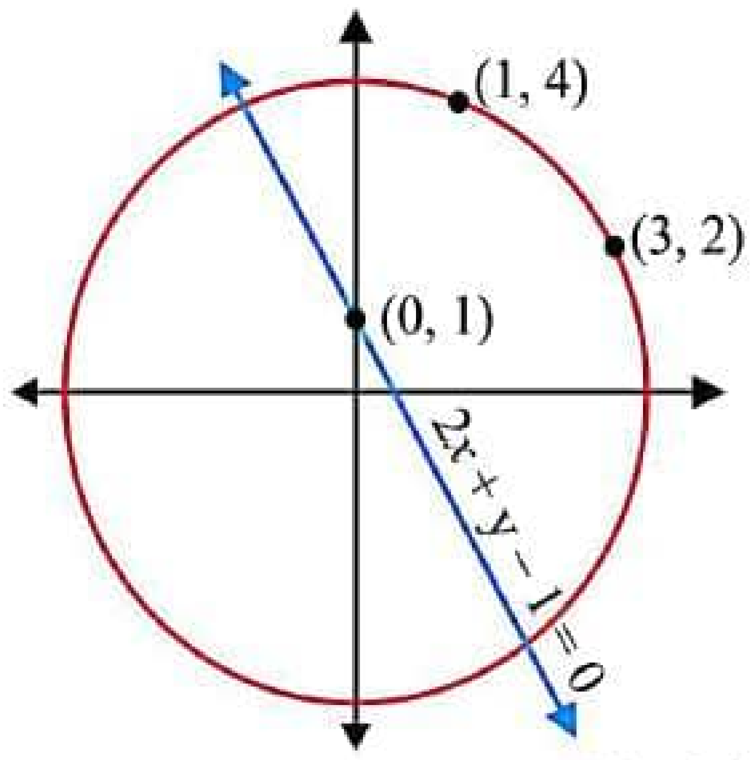


Fig. 8.8

Subtracting equation (iii) from equation (ii)

We get $\quad 4 - 4g + 4f = 0$

or $\quad 1 - g + f = 0 \quad \ldots(\text{v})$

Solving equation (iv) and (v) simultaneously,

We get $\quad g = 0$ and $f = -1$

By using $\quad g = 0$ and $f = -1$, equation (ii) becomes

$17 - 8 + c = 0$

$\Rightarrow \quad c = -9$

By using these values of g, f and c in equation (i)

we get $\quad x^2 + y^2 - 2y - 9 = 0$

(c) Equation of circle passing through two points and equation of tangent at one of these points is known

In the following example, the method of finding equation of circle is explained when the circle passes through two points and equation of tangent at one of these points is known.

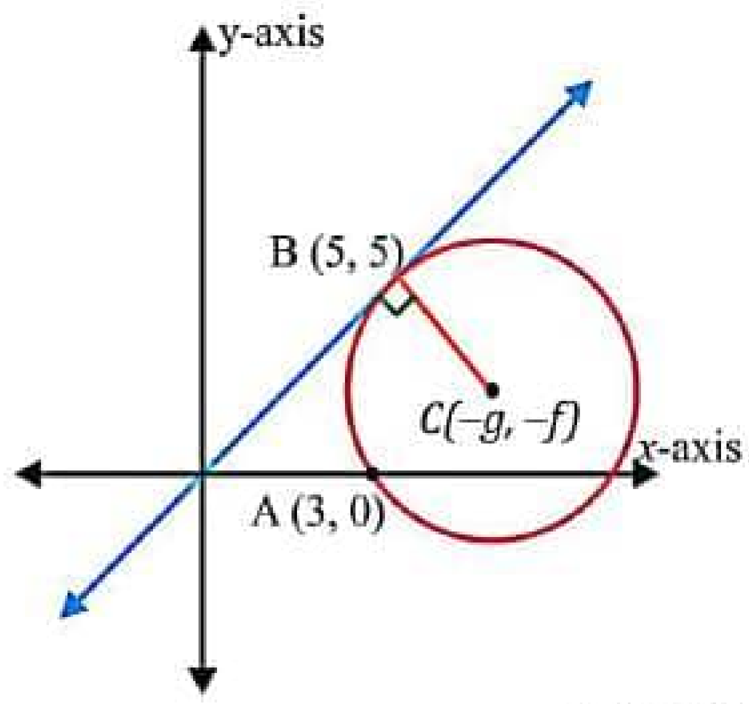


Fig. 8.9

Example: Find the equation of circle passing through $A(3,0)$ and $B(5,5)$, whereas the line $x - y = 0$ is tangent to the circle at B.

Solution: Let the general equation of circle with centre $C(-g,-f)$ be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots(\text{i})$$

$\because \quad$ The circle passes through $(5,5)$ and $(3,0)$

$\therefore \quad$ we have $\quad 50 + 10g + 10f + c = 0 \quad \ldots(\text{ii})$

and $\quad 9 + 6g + c = 0 \quad \ldots(\text{iii})$

Subtracting equation (iii) from equation (ii)

we get $\quad 41 + 4g + 10f = 0 \quad \ldots(\text{iv})$

$\because \quad$ radial segment BC and the given tangent are perpendicular

$\therefore \quad$ product of their slopes is -1

i.e., $\quad \dfrac{5+f}{5+g} = -1 \quad$ (where slope of $\overline{BC} = \dfrac{5+f}{5+g}$ and slope of tangent $= 1$)

$\Rightarrow \quad 5 + f = -5 - g$

$$\Rightarrow \quad f + g + 10 = 0 \qquad \ldots(v)$$

Solving equation (iv) and equation (v) simultaneously,

we get $\quad g = -\frac{59}{6}$ and $f = -\frac{1}{6}$

By using $\quad g = -\frac{59}{6}$ in equation (iii)

we get $\quad c = 50$

By subtracting values of g, f and c in equation (i), we get

$$x^2 + y^2 - \frac{59x}{3} - \frac{y}{3} + 50 = 0$$

or $$3x^2 + 3y^2 - 59x - y + 150 = 0$$

(d) Equation of circle passing through two points and touching a given line.

In this case the given tangent does not pass through any of the two given points of the circle. The method is explained in the following example.

Example 1. Find the equation of circle passing through two points $(1, 0)$ and $(0, 1)$ and touches the line $x + y = 0$.

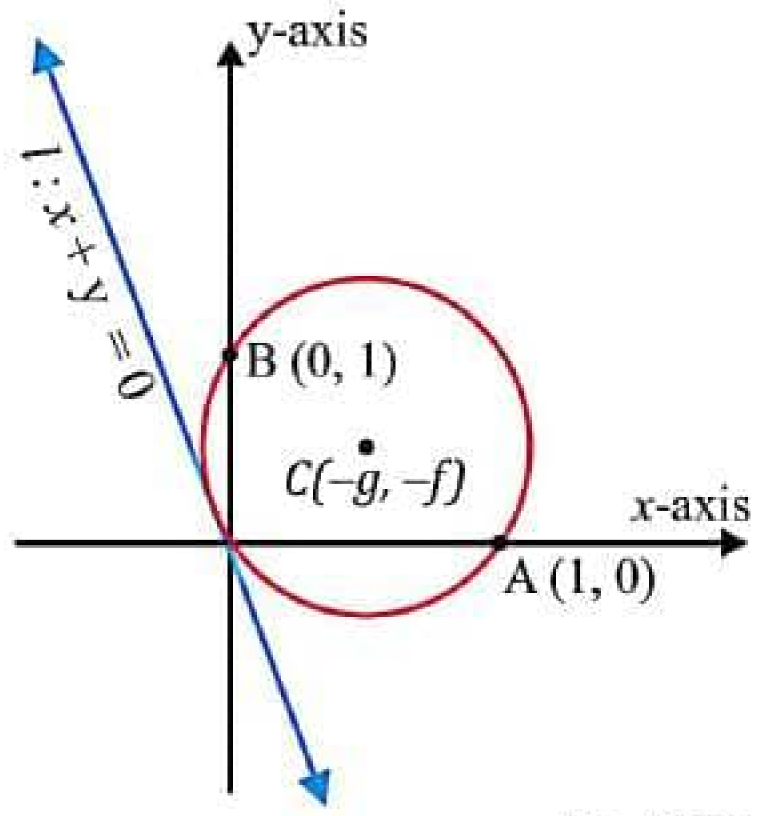


Fig. 8.10

Solution: Since line is touches the circle S at origin $(0, 0)$

$\therefore \quad c = 0$

Let $x^2 + y^2 + 2gx + 2fy = 0$ be the desire equation of a circle.

Since points $(1, 0)$ and $(0, 1)$ are the points in the circle.

$\therefore \quad 1 + 0 + 2g + 0 = 0 \Rightarrow g = -\frac{1}{2}$

and $\quad 0 + 1 + 2f + 0 = 0 \Rightarrow g = -\frac{1}{2}$

Thus, required equation of a circle is

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\left(-\frac{1}{2}\right)y = 0$$

$$\Rightarrow \quad \boxed{x^2 + y^2 - x - y = 0}$$

Example 2. Find the equation of circle containing the points $(-2, 1)$ and $(-4, 3)$ and touching y-axis.

Solution: Let equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$$

$\because \quad$ circle passes through $(-2, 1)$ and $(-4, 3)$

$\therefore \quad$ we have

$$5 - 4g + 2f + c = 0 \qquad \ldots(ii)$$

and $25 - 8g + 6f + c = 0$ ...(iii)

$\because$ circle touches y-axis.

$\therefore$ radius of circle = modulus of abscissa of centre

i.e., $\sqrt{g^2 + f^2 - c} = |-g|$

squaring both sides

we get $c = f^2$

Multiplying equation (ii) by 2

and using $c = f^2$ in the resultant equation and equation (iii)

we get $10 - 8g + 4f + 2f^2 = 0$ ...(iv)

$25 - 8g + 6f + f^2 = 0$ ...(v)

Subtracting equation (v) from equation (iv)

we get $-15 - 2f + f^2 = 0$

or $f^2 - 2f - 15 = 0$

$\Rightarrow$ $(f - 5)(f + 3) = 0$

$\Rightarrow$ $f = 5$ and $f = -3$

So, $c = 25$ and $c = 9$

If $f = 5$ and $c = 25$ then equation (ii)

becomes $5 - 4g + 10 + 25 = 0$

$\Rightarrow$ $4g = 40$

or $g = 10$

If $f = -3$ and $c = 9$ then equation (ii) becomes

$5 - 4g - 6 + 9 = 0$

$\Rightarrow$ $4g = 8$

$\Rightarrow$ $g = 2$

Now, if $f = 5, c = 25$ and $g = 10$ then equation (i), becomes

$x^2 + y^2 + 20x + 10y + 25 = 0$

and if $f = -3, c = 9$ and $g = 2$

then equation (i) becomes

$x^2 + y^2 + 4x - 6y + 9 = 0$

Exercise 8.2

1. Find the equation of the circle through the given points.

(i) $(0, 0), (0, 3), (-4, 0)$ (ii) $(0, 10), (-10, 0), (8, 6)$

(iii) $(0, 3), (2, -1), (1, 0)$ (iv) $(7, -3), (-7, 5), (11, 5)$

(v) $(1, 1), (2, -1), (3, -2)$

2. Find the equation of circle through the points $(1,2),(2,3)$ and having centre on
   (i) x-axis (ii) y-axis.
3. Find the equation of circle through the points $(3,1),(2,2)$ and having centre on the line $x+y-3=0$.
4. Find the equation of the circle through the points $(0,-1)$ and $(3,0)$ and the line $3x+y-9=0$ is tangent to it at $(3,0)$.
5. Find the equation of the circle through origin with x-intercept 2 and is tangent to the line $y-1=0$.
6. Find the equation of a circle containing the points $(1,2),(2,3)$ and having centre on $x-y+1=0$.
7. Find the equation of the circum-circle of the triangle with vertices $(1,-2),(-5,2)$ and $(3,4)$.
8. Find the equation of circle containing the points $(1,-2)$ and $(3,-4)$ and touching x-axis.
9. Find the equation of circle containing the points $(6,0)$ and touching the line $x=y$ at $(4,4)$.
10. Show that the equation of circle with centre $(-g,-f)$ and;
    (i) touching x-axis is of the form $x^2+y^2+2gx+2fy+g^2=0$
    (ii) touching y-axis is of the form $x^2+y^2+2gx+2fy+f^2=0$
11. Find the equation of circle passing through origin and having intercepts 6 and 8.
12. Find the equation of the circle which passes through the two points $(b,0)$ and $(-b,0)$ and whose radius is a unit.
13. Find the equation of the circle which passes through the point $(5,0)$ and $(0,-5)$ and whose radius is 5 unit.

8.5 Tangent and Normal

In this section, we will discuss about the conditions of tangency and normality of a line to a circle along with their equations. But, first let us revise the concepts of secant, tangent and normal.

In geometry, a line which touches a curve at a single point is tangent to the curve and the point called point of tangency or point of contact.

Any line which is perpendicular to the tangent at the point of tangency is called normal to the curve whereas, secant is the line which intersects a curve at a minimum of two distinct points as shown in Fig. 8.11.

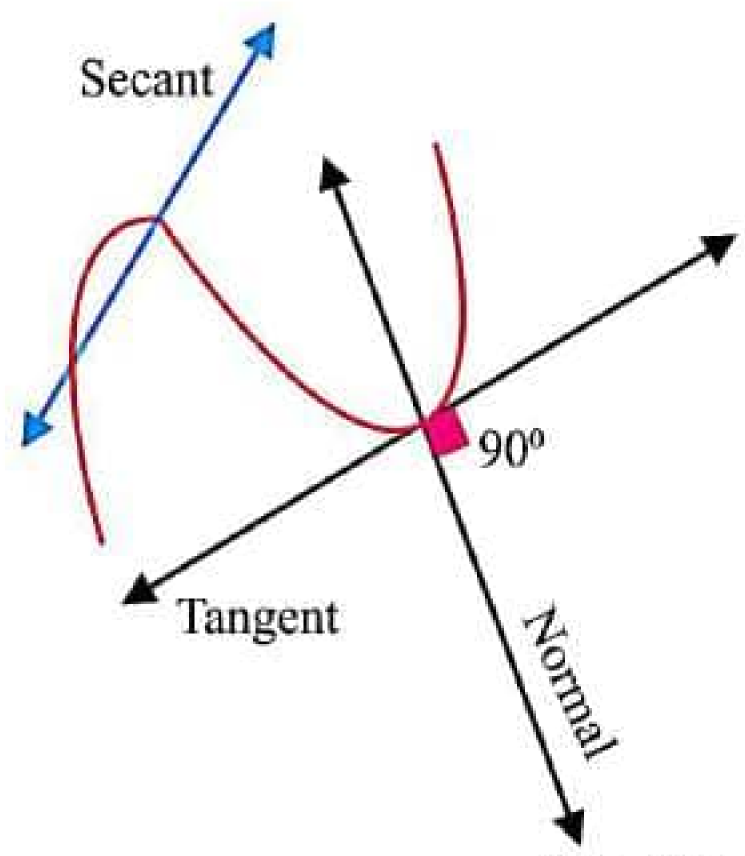


Fig. 8.11

In case of circle, secant intersects the circle at exactly two points. In the figure 8.12 line l is secant to the circle through two points P and Q. If P gets closer to Q and ultimately becomes coincident with Q, the secant l becomes tangent to the circle at point Q. In circle, normal always passes through the centre of the circle.

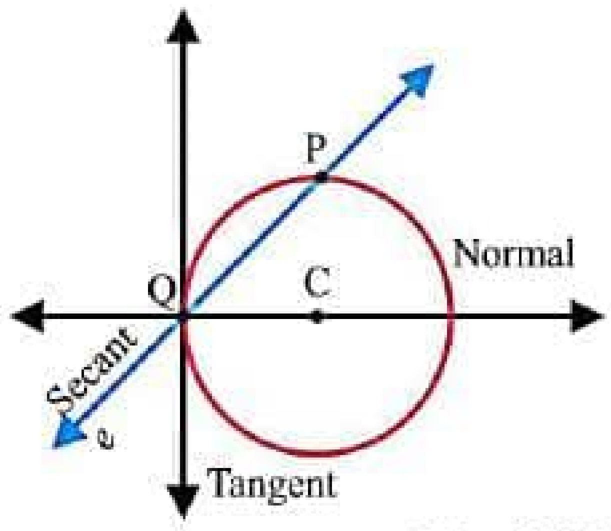


Fig. 8.12

8.5.1 Find the condition when a line intersects the circle

We are aware of the fact that a line can cut or touch a circle and sometimes, it neither cuts nor touches the circle at a point. In this section we will discuss these conditions in detail.

Consider a line $y = mx + c$ and the circle $x^2 + y^2 = r^2$.

Solving both equations simultaneously,

we get

$$x^2 + (mx + c)^2 = r^2$$

$$\Rightarrow \quad x^2 + m^2x^2 + 2mcx + c^2 = r^2$$

$$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0 \quad \ldots\text{(i)}$$

Since roots of equation (i) represent abscissas of the points of intersection A and B of the given circle and line as shown in the figure 8.13.

Therefore, nature of roots of the quadratic equation (i) will represent the nature of parallel lines l_1, l_2 and l_3, each of the slope m, with respect to the given circle.

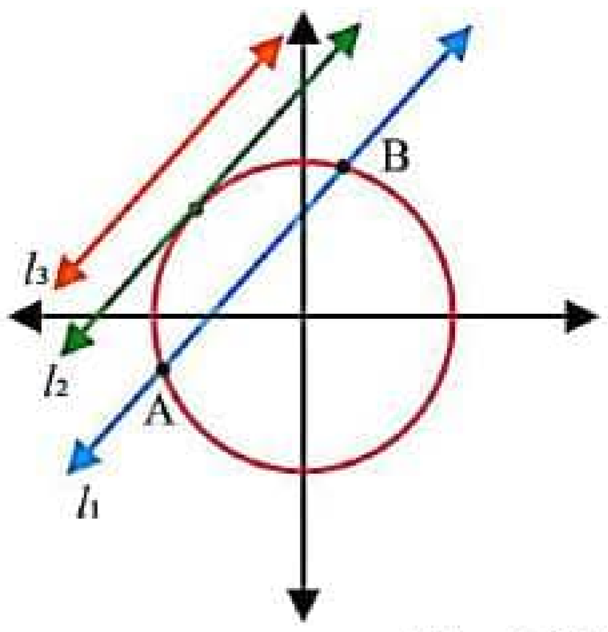


Fig. 8.13

Here discriminant of equation (i) is:

$$\Delta = (2mc)^2 - 4(1 + m^2)(c^2 - r^2)$$

or $$\Delta = 4m^2c^2 - 4c^2 + 4r^2 - 4m^2c^2 + 4m^2r^2$$

$$\Rightarrow \quad \Delta = 4\{r^2(1 + m^2) - c^2\}$$

We know that the roots of equation (i) will be real and unequal if $\Delta > 0$.

i.e., $$r^2(1 + m^2) - c^2 > 0$$

$$\Rightarrow \quad r^2(1 + m^2) > c^2 \quad \ldots\text{(ii)}$$

This is the condition when points of intersection are real and distinct. This value of c^2 corresponds to l_1 which intersects the circle at two real and distinct points. Condition (ii) is the condition when line intersects the circle i.e., condition of secancy.

We know that

the roots of equation (i) will be real and equal.

If $\Delta = 0$

i.e., $$r^2(1 + m^2) - c^2 = 0$$

$$\Rightarrow \quad \boxed{r^2(1+m^2) = c^2} \qquad \ldots\text{(iii)}$$

This is the condition when points of intersection are real and coincident. This value of c^2 corresponds to l_2 which touches the given circle at a single point.

Hence condition (iii) is the condition of tangency of the line. We also know that the roots of equation (i) will be non-real if $\Delta < 0$.

i.e., $\quad r^2(1+m^2) - c^2 < 0$

$$\Rightarrow \quad \boxed{r^2(1+m^2) < c^2} \qquad \ldots\text{(iv)}$$

This is the condition when points of intersection are imaginary. This value of c^2 corresponds to l_3 which neither cuts nor touches the given circle. Hence condition (iv) is the condition when the line is neither secant nor tangent.

8.5.2 Find the condition when a line touches the circle

As discussed in section 8.5.1 a line $y = mx + c$ will be tangent to the circle $x^2 + y^2 = r^2$ if $c^2 = r^2(1+m^2)$.

In general, a line l will be tangent to any given circle if distance of the line from centre is always equal to the radius of the circle. So, in order to find the condition of tangency of line to the given circle, we equate the distance of the line from the centre of the circle and radius of the given circle. (Fig. 8.14)

Alternatively, we take discriminant of the quadratic equation as zero which is obtained by solving equations of given circle and line simultaneously as we did in section 8.5.1.

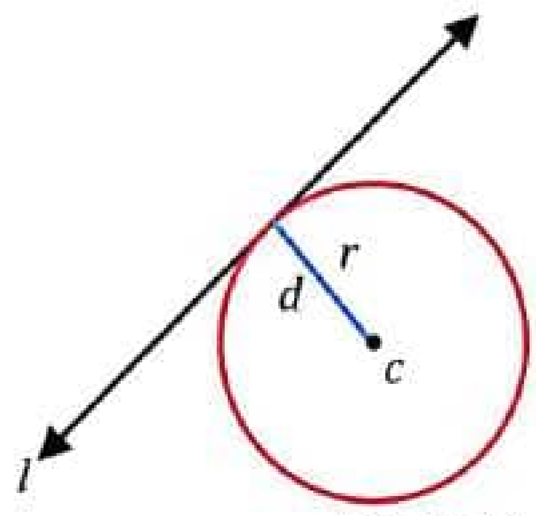


Fig. 8.14

Condition of tangency of a line $lx + my + n = 0$ to the circle $x^2 + y^2 = r^2$.

$\because \quad$ centre of the circle $x^2 + y^2 = r^2$ is origin. (Fig. 8.15)

$\therefore \quad$ distance of line: $lx + my + n = 0$ from centre is:

$$d = \left|\frac{n}{\sqrt{l^2+m^2}}\right|$$

Now, given line will be tangent to the given circle.

if $\quad d =$ radius of circle

i.e., $\quad \left|\frac{n}{\sqrt{l^2+m^2}}\right| = r$

Squaring both sides

$$\frac{n^2}{l^2+m^2} = r^2$$

$$\Rightarrow \quad \boxed{n^2 = r^2(l^2+m^2)}$$

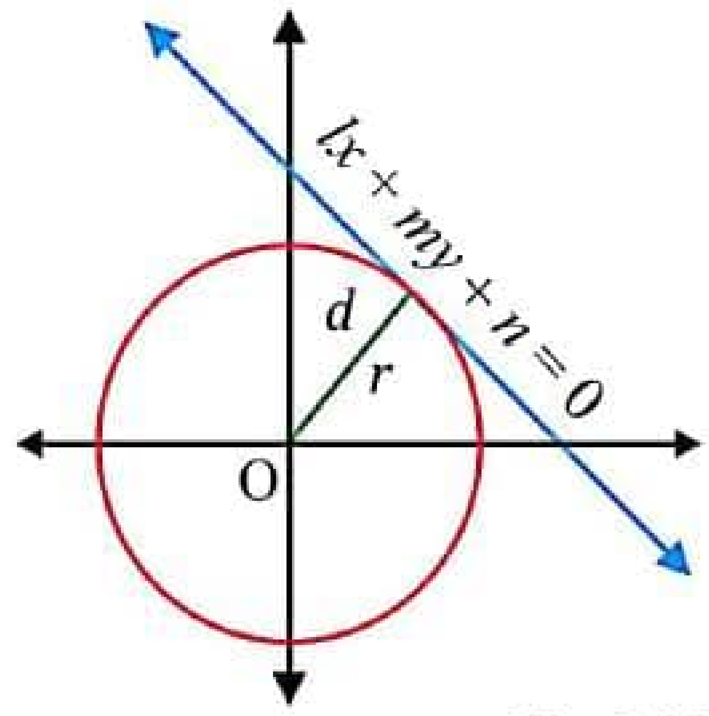


Fig. 8.15

This is the condition when line $lx + my + n = 0$ will touch the circle $x^2 + y^2 = r^2$.

Alternative Method

We have, equation of line

$$lx + my + n = 0 \quad \dots(i)$$

and equation of circle.

$$x^2 + y^2 = r^2 \quad \dots(ii)$$

Solving (i) and (ii), simultaneously, we get

$$x^2 + \left(\frac{-lx - n}{m}\right)^2 = r^2$$

$\Rightarrow \quad m^2x^2 + l^2x^2 + 2nlx + n^2 = m^2r^2$

$\Rightarrow \quad (l^2 + m^2)x^2 + 2nlx + n^2 - m^2r^2 = 0 \quad \dots(iii)$

Given line will be tangent to the given circle

if Discriminant of equation (iii) vanishes

i.e., $4n^2l^2 - 4(l^2 + m^2)(n^2 - m^2r^2) = 0$

$\Rightarrow \quad 4n^2l^2 - 4n^2l^2 + 4l^2m^2r^2 - 4m^2n^2 + 4m^4r^2 = 0$

$\Rightarrow \quad 4m^2(l^2r^2 - n^2 + m^2r^2) = 0$

$\Rightarrow \quad l^2r^2 - n^2 + m^2r^2 = 0 \quad$ (Let $m \neq 0$)

$\Rightarrow \quad n^2 = r^2(l^2 + m^2)$

This is the condition of tangency of given line $lx + my + n = 0$ to the given circle $x^2 + y^2 = r^2$.

Example 1. Find the condition of tangency and secancy of the line $y = mx + k$ with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Solution: We have the line $mx - y + k = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

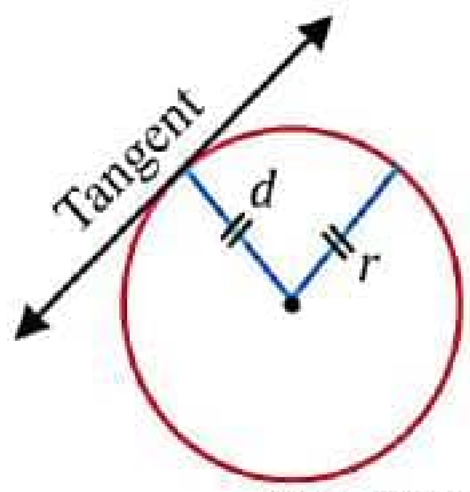


Fig. 8.16

Given line will be tangent to the circle, if distance of line from the centre of circle is equal to the radius of the circle, i.e., $d = r$ as shown in Fig. 8.16.

or $$\left|\frac{-mg+f+k}{\sqrt{m^2+1}}\right| = \sqrt{g^2 + f^2 - c}$$

squaring both sides

$$(f + k - mg)^2 = (g^2 + f^2 - c)(m^2 + 1)$$

This is the required condition of tangency.

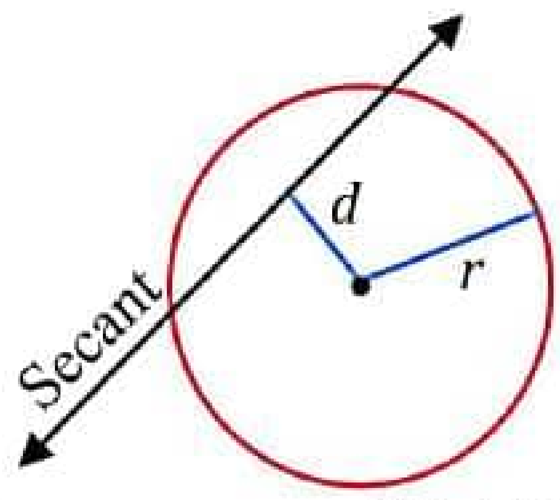


Fig. 8.17

We know that given line will be secant to the circle if, distance of line from the centre is less than the radius of the circle as shown in Fig. 8.17.

i.e., $d < r$

or $\left|\frac{-mg+f+k}{\sqrt{m^2+1}}\right| < \sqrt{g^2+f^2-c}$

This is the required condition of secancy.

Example 2. Find the value of r when the line $x = 2y + 4$ should be:

(i) a tangent to the circle $x^2 + y^2 = r^2$

(ii) a secant to the circle $x^2 + y^2 = r^2$

Solution: (i) value of r when line is tangent

We have, line: $x = 2y + 4$ ...(i)

and circle: $x^2 + y^2 = r^2$ ...(ii)

solving equations (i) and (ii)

we get $(2y + 4)^2 + y^2 = r^2$

$\Rightarrow \quad 5y^2 + 16y + 16 - r^2 = 0$ ...(iii)

Given line will be tangent to the circle,

if $\Delta = 0$

i.e., $(16)^2 - 4(5)(16 - r^2) = 0$

$\Rightarrow \quad 256 - 320 + 20r^2 = 0$

$\Rightarrow \quad 20r^2 = 64$

$\Rightarrow \quad r = \pm\frac{4}{\sqrt{5}}$

(ii) Value of r when given line is secant

From equation (iii)

$$\Delta = 256 - 20(16 - r^2)$$
$$= 20r^2 - 64$$

We know that given line will be secant to the circle,

if $\Delta > 0$

i.e., $20r^2 - 64 > 0$

$\Rightarrow \quad r^2 > \frac{16}{5}$

$\Rightarrow \quad r > \frac{16}{5}$ or $r < -\frac{16}{5}$

8.5.3 Find the equation of a tangent to a circle in slope form:

Let m be the slope of a line which is tangent to the circle $x^2 + y^2 = r^2$ then the equation of tangent will be

$y = mx + c$ ...(i)

where c is the y-intercept of the tangent.

According to the condition of tangency

$$c^2 = r^2(1+m^2)$$

$$\Rightarrow \quad c = \pm r\sqrt{1+m^2}$$

By using value of c in equation (i)

we get $\boxed{y = mx \pm r\sqrt{1+m^2}}$

This is the equation of tangent to the circle $x^2+y^2=r^2$ in slope form.

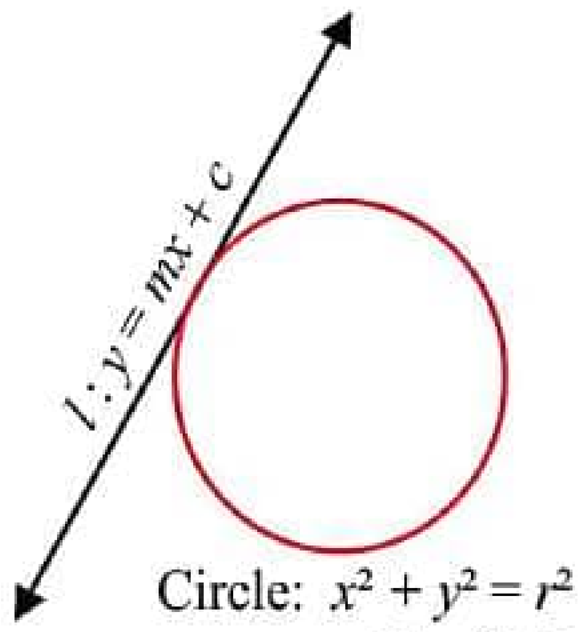


Fig. 8.18

Example: Find the equation of tangent to $x^2+y^2=25$ with the slope 2.

Solution:

Here, slope of tangent $= m = 2$

and radius $= r = 5$

We know that the equation of tangent to the given circle will be

$$y = mx \pm r\sqrt{1+m^2}$$

By using values of m and r

we get $y = 2x \pm 5\sqrt{5}$

So, there will be two tangents to the given circle with slope 2 which are

$$y = 2x + 5\sqrt{5} \text{ and } y = 2x - 5\sqrt{5}$$

8.5.4 Find the equations of a tangent and a normal to a circle at a point

Equation of tangent and normal to a circle at a given point

Let the line l be the tangent to the circle

$x^2+y^2+2gx+2fy+c=0$ at the given point (x_1, y_1) as shown in the figure 8.19.

$\because$ tangent to the circle is perpendicular to the radial segment at the point of contact i.e., $l \perp \overline{CP}$

$$\therefore \quad \text{slope of tangent} = m = -\frac{1}{\text{slope of } \overline{CP}}$$

$$= -\frac{1}{\frac{y_1+f}{x_1+g}}$$

$$= -\frac{x_1+g}{y_1+f}$$

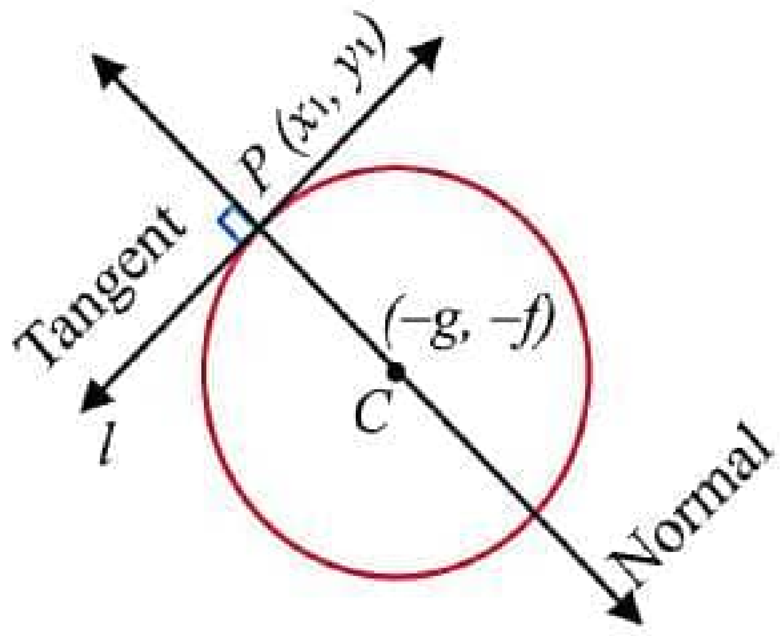


Fig. 8.19

By point-slope form the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$ ...(i)

$\Rightarrow$ $(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$

or $yy_1 + fy - y_1^2 - fy_1 + xx_1 + gx - x_1^2 - gx_1 = 0$

$\Rightarrow$ $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$ ...(ii)

$\because$ (x_1, y_1) lies on the given circle

$\therefore$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c$

So, equation (ii) becomes

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ...(iii)

This is the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1).

In case centre of circle is at origin then $g = f = 0$

So, equation (iii) becomes

$xx_1 + yy_1 + c = 0$ ...(iv)

We know that

radius of circle $= r = \sqrt{g^2 + f^2 - c}$

i.e., $c = g^2 + f^2 - r^2$

$\Rightarrow$ $c = -r^2$ $(\because \ g = f = 0)$

So, equation (iv) becomes

$xx_1 + yy_1 - r^2 = 0$

i.e., $xx_1 + yy_1 = r^2$ ...(v)

This is the equation of tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1).

We know that normal is perpendicular to the tangent at the point of contact.

So, slope of normal $= m' = -\frac{1}{m}$

$= \frac{y_1 + f}{x_1 + g}$

Now, by point-slope form, the equation of normal will be

$y - y_1 = m'(x - x_1)$

i.e., $y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1)$ ...(vi)

This is the equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1).

In case centre is at origin then $g = f = 0$.

So, equation (vi) becomes

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$\Rightarrow \quad x_1y - x_1y_1 = xy_1 - x_1y_1$

$\Rightarrow \quad x_1y - xy_1 = 0$

This is the equation of normal to the circle $x^2 + y^2 = r^2$ at a given point (x_1, y_1).

- **Equation of a tangent and a normal to a circle at a given point using derivative.**

We know that derivative $\left(\frac{dy}{dx}\right)$ at a point (x_1, y_1) of a curve $y = f(x)$ is the slope of the tangent to the curve at that point.

Thus, the equation of tangent to any circle at the point (x_1, y_1) is

$y - y_1 = m(x - x_1)$

where $m = \frac{dy}{dx}$ at (x_1, y_1)

Since normal is perpendicular to the tangent at the point (x_1, y_1)

Therefore, equation of the normal to any circle will be

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Example 1. Find the tangents and normals to the following circles without using derivatives.

(i) $x^2 + y^2 = 25$ at $(3, 4)$

(ii) $x^2 + y^2 + 6x + 4y = 132$ at $(6, 6)$

Solution: (i) $\mathbf{x^2 + y^2 = 25}$ **at** $\mathbf{(3, 4)}$

Here $(x_1, y_1) = (3, 4)$

and $r = 5$

Now, equation of tangent to the given circle will be

$xx_1 + yy_1 = r^2$

i.e., $3x + 4y = 25$

Also, the equation of normal to the given circle will be

$x_1y - xy_1 = 0$

i.e., $3y - 4x = 0$

(ii) $\mathbf{x^2 + y^2 + 6x + 4y = 132}$ **at** $\mathbf{(6, 6)}$

Given circle is $x^2 + y^2 + 6x + 4y - 132 = 0$

Comparing with general equation of circle

we get $2g = 6 \quad \Rightarrow \quad g = 3$;

$2f = 4 \quad \Rightarrow \quad f = 2$

and $\quad c = -132$

We also have $(x_1, y_1) = (6, 6)$

Now, equation of tangent to the given circle will be

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Using values, we get

$$6x + 6y + 3(x + 6) + 2(y + 6) - 132 = 0$$

Also, the equation of normal to the given circle will be

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

i.e., $\quad y - 6 = \frac{6+2}{6+3}(x - 6)$

$\Rightarrow \quad y - 6 = \frac{8}{9}(x - 6)$

$\Rightarrow \quad 9y - 54 = 8x - 48$

$\Rightarrow \quad 9x - 9y = 6 = 0$

Example 2. Find the equation of tangent and normal to $x^2 + y^2 = 100$ at $(6, 8)$.

Solution: We have circle,

$$x^2 + y^2 = 100$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

So, slope of tangent to the given circle at $(6, 8) = \left(\frac{dy}{dx}\right)_{(6,8)}$

$$= -\frac{6}{8}$$

$$= -\frac{3}{4} = m$$

We also have $(x_1, y_1) = (6, 8)$

By point-slope form the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $\quad y - 8 = -\frac{3}{4}(x - 6)$

$\Rightarrow \quad 4y - 32 = -3x + 18$

$\Rightarrow \quad 3x + 4y - 50 = 0$

$\because$ Normal is perpendicular to the tangent at $(6, 8)$

$\therefore$ its slope $= m' = -\dfrac{1}{m}$

$= \dfrac{4}{3}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - 8 = \frac{4}{3}(x - 6)$

$\Rightarrow$ $3y - 24 = 4x - 24$

$\Rightarrow$ $4x - 3y = 0$

8.5.5 Find the length of tangent to a circle from a given external point

Let $P(x, y)$ be the point of contact of the tangent from the external point $E(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre is $C(-g, -f)$. $\overline{EP}$ is called tangent segment and its length is called length of tangent.

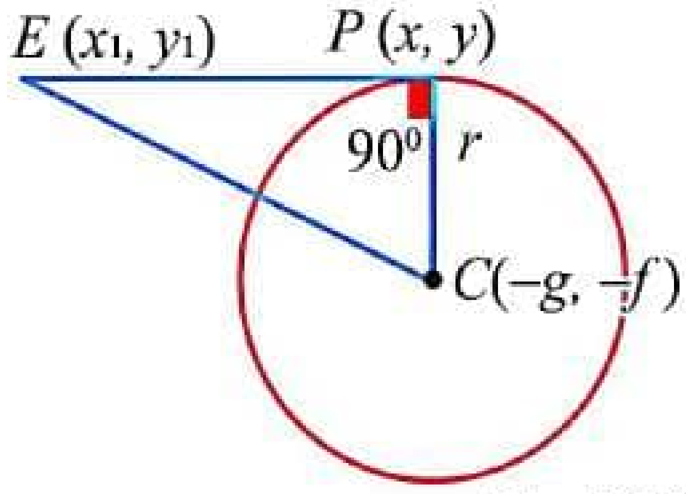


Fig. 8.20

$\because$ $\overline{EP} \perp \overline{CP}$

$\therefore$ CEP is a right angled triangle as shown in the figure 8.20.

In ΔCEP, by Pythagoras theorem

$$|\overline{CE}|^2 = |\overline{CP}|^2 + |\overline{EP}|^2$$

i.e., $(x_1 + g)^2 + (y_1 + f)^2 = r^2 + |\overline{EP}|^2$

$\Rightarrow$ $|\overline{EP}|^2 = x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - (g^2 + f^2 - c)$

$\left(\because \quad r = \sqrt{g^2 + f^2 - c}\right)$

So, $|\overline{EP}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ ...(i)

So, the length of tangent from (x_1, y_1) to the circle in general form is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Example: Find the length of tangent from $(-2, 3)$ to the circle $x^2 + y^2 - 5x - 2y + 1 = 0$.

Solution: Given circle is $x^2 + y^2 - 5x - 2y + 1 = 0$

Here $2g = -5$,

$2f = -2$,

$c = 1$ and $(x_1, y_1) = (-2, 3)$

Now,

length of tangent $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{(-2)^2 + (3)^2 - 5(-2) - 2(3) + 1}$

$= \sqrt{4 + 9 + 10 - 6 + 1}$

$= \sqrt{18}$ units.

8.5.6 Prove that two tangents drawn to a circle from an external point are equal in length

Let $\overrightarrow{PA}$ and $\overrightarrow{PB}$ be two tangents to the given circle $x^2+y^2+2gx+2fy+c=0$ with centre $C(-g,-f)$ from an external point $P(x_1,y_1)$ as shown in the figure 8.21.

We know that a tangent to the circle is perpendicular to its radial segment at the point of contact.

So, we have two right triangles PAC and PBC with right angles at A and B respectively.

In right ΔPAC, by using Pythagoras theorem

$$|\overline{CP}|^2 = |\overline{AC}|^2 + |\overline{AP}|^2$$

$$\Rightarrow \quad (x_1+g)^2 + (y_1+f)^2 = r^2 + |\overline{AP}|^2$$

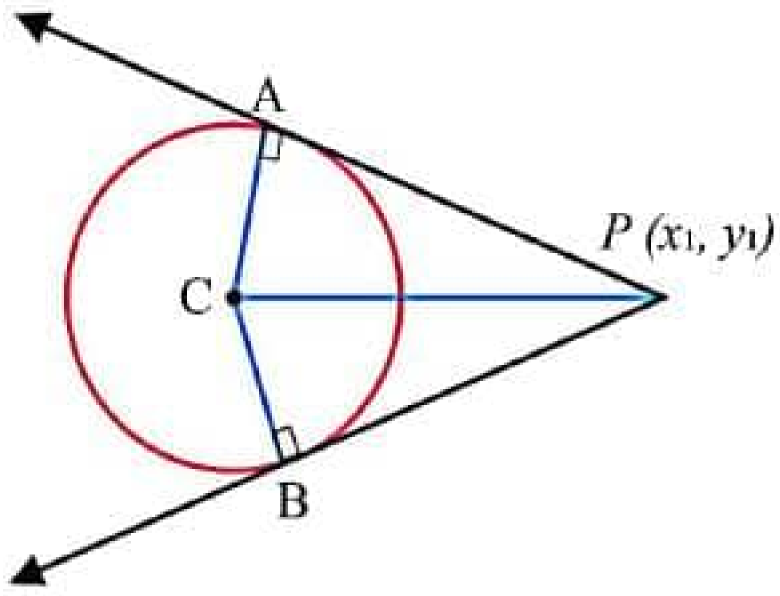


Fig. 8.21

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + 2gx_1 + g^2 + y_1^2 + 2gy_1 + f^2 - r^2$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + g^2 + f^2 - g^2 - f^2 + c$$

$$\left(\because \quad r = \sqrt{g^2+f^2-c}\right)$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$\Rightarrow \quad |\overline{AP}|^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

So, $\quad |\overline{AP}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \qquad \ldots(i)$

Similarly, in right ΔPBC

$$|\overline{PB}| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$|\overline{AP}| = |\overline{PB}|$$

Hence two tangents drawn to a circle from an external point are equal in length.

Exercise 8.3

1. Check whether the following lines are tangent, secant or neither to the circle $x^2+y^2=25$.
   (i) $y = x + 3$
   (ii) $y = \sqrt{3}x + 10$
   (iii) $y = 2x + 15$
2. Find the condition of tangency and secancy of the line $y = 2x + k$ with the circle $x^2 + y^2 + 10x + 20y + c = 0$.
3. Find the equation of tangent to $x^2 + y^2 = 36$ with the slope $\sqrt{3}$.

4. Find the equation of tangent and normal:
   (i) at $(1,-4)$ to the circle $x^2+y^2=17$
   (ii) at $(4,1)$ to the circle $x^2+y^2-4x+2y=3$
5. Find the length of tangent:
   (i) from $(6,1)$ to the circle $x^2+y^2=4$
   (ii) from $(2,5)$ to the circle $x^2+y^2+8x-5y=7$.
6. Find the condition that the line $y=mx+c$ may be tangent to the circle $(x-h)^2+(y-k)^2=r^2$.
7. Show that circles $x^2+y^2-6x-6y+10=0$ and $x^2+y^2=2$ touch each other and find the point of contact.
8. Find the equation of tangent (s) to the circle $x^2+y^2=25$.
   (i) at the point whose abscissa is 3.
   (ii) at the point whose ordinate is -4.
   (iii) which is parallel to $3x+4y+1=0$
   (iv) which is perpendicular to $3x+4y+1=0$
9. Find the equations of tangents to $x^2+y^2-6x-2y+9=0$ through origin. Find also their respective points of contact.
10. Show that the line $ax+by+al+bm=0$ is normal to the circle $x^2+y^2+2lx+2my+c=0$ for all values of a and b.
11. Find (i) the product of abscissa (ii) the product of ordinates of the points, where the line $y=mx$ meets the circle $x^2+y^2+2gx+2fy+c=0$.
12. Prove that the line $y=x+k\sqrt{2}$ touches the circle $x^2+y^2=k^2$ and find its point of contact.
13. Find the condition that the line $3x+4y=c$ may touch the circle $x^2+y^2=8x$.
14. Find whether the line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y-1=0$.
15. Prove that the two circles $x^2+y^2+2gx+c=0$ and $x^2+y^2+2fy+c=0$ touch each other, if $\frac{1}{f^2}+\frac{1}{g^2}=\frac{1}{c}$.

8.6 Properties of Circle

In this section we will prove some theorem of Euclidean geometry analytically which are related to the circle.

Prove analytically the following properties of a circle.

- **Perpendicular from the centre of a circle on a chord bisects the chord.**

Let $\overline{AB}$ be a chord of circle $x^2+y^2+2gx+2fy+c=0$ with centre $C(-g,-f)$. where the end points of the chord are $A(x_1,y_1)$ and $B(x_2,y_2)$ as shown in the Fig. 8.22.

Furthermore $\overline{CD}$ is perpendicular from centre $\overline{CD}$ to the chord $\overline{AB}$.

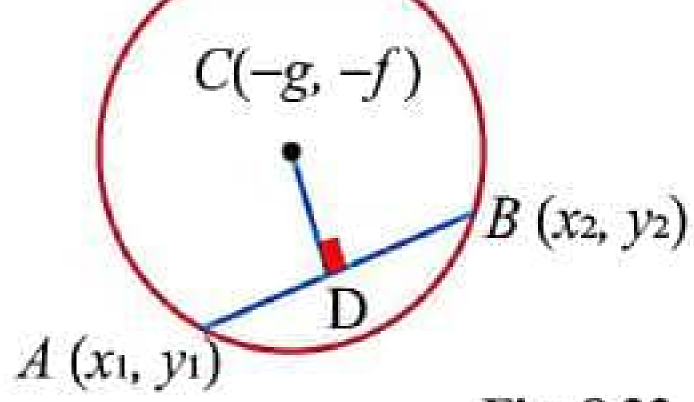


Fig. 8.22

$\because$ $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the circle

$\therefore$ we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(i)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{...(ii)}$$

Subtracting equation (ii) from equation (i), we get

$$(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + 2g(x_1 - x_2) + 2f(y_1 - y_2) = 0$$

$$\Rightarrow \boxed{2g(x_1 - x_2) + 2f(y_1 - y_2) = -x_1^2 + x_2^2 - y_1^2 + y_2^2} \quad \text{...(iii)}$$

Now, slope of $\overline{CD} = -\dfrac{1}{\frac{y_2-y_1}{x_2-x_1}}$

or $m = -\dfrac{(x_1-x_2)}{y_1-y_2}$

Now, equation of $\overline{CD}$, by point-slope form will be

$$y + f = m(x + g)$$

i.e., $$y + f = -\frac{(x_1-x_2)}{y_1-y_2}(x + g)$$

$$\Rightarrow (y_1 - y_2)(y + f) = -(x_1 - x_2)(x + g)$$

$$\Rightarrow y(y_1 - y_2) + f(y_1 - y_2) = -(x_1 - x_2)x - g(x_1 - x_2)$$

$$\Rightarrow g(x_1 - x_2) + f(y_1 - y_2) = -(x_1 - x_2)x - y(y_1 - y_2)$$

Using equation (iii), we get

$$2x(x_1 - x_2) + 2y(y_1 - y_2) = x_1^2 - x_2^2 + y_1^2 - y_2^2 \quad \text{...(iv)}$$

This is the equation of perpendicular $\overline{CD}$ from centre of circle to the chord $\overline{AB}$.

Now, midpoint of $\overline{AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

By substituting midpoint in equation (iv), we get

$$2\left(\frac{x_1 + x_2}{2}\right)(x_1 - x_2) + 2\left(\frac{y_1 + y_2}{2}\right)(y_1 - y_2) = x_1^2 - x_2^2 + y_1^2 - y_2^2$$

$$\Rightarrow x_1^2 - x_2^2 + y_1^2 - y_2^2 = x_1^2 - x_2^2 + y_1^2 - y_2^2$$

$\because$ Midpoint of $\overline{AB}$ satisfies equation of perpendicular $\overline{CD}$

$\therefore$ $\overline{CD}$ bisects the chord $\overline{AB}$.

- **Perpendicular bisector of any chord of a circle passes through the centre of the circle.**

  Let $\overline{AB}$ be a chord of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $C(-g, -f)$.

  Let l be the perpendicular bisector of $\overline{AB}$ which cuts $\overline{AB}$ at midpoint D whereas the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively as shown in Fig. 8.23.

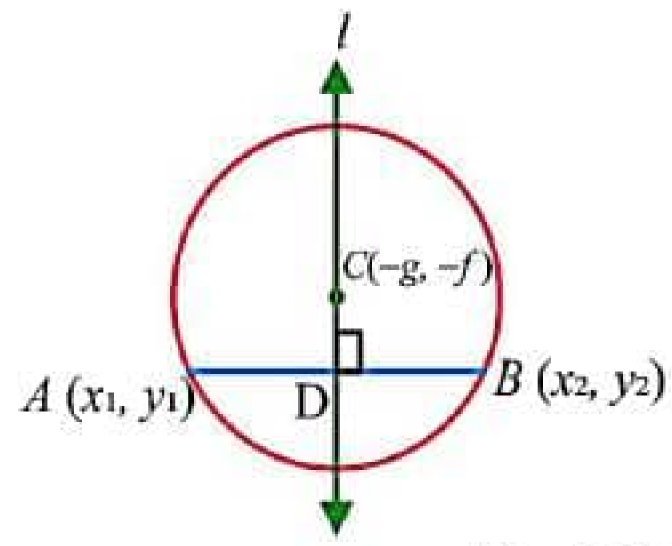


Fig. 8.23

Here midpoint $D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

and slope of $\overline{AB} = \frac{y_2-y_1}{x_2-x_1}$

$\because$ l is perpendicular on $\overline{AB}$

$\therefore$ slope of $l = -\frac{(x_2-x_1)}{y_2-y_1}$

Now, equation of l will be

$$\left\{y - \left(\frac{y_1+y_2}{2}\right)\right\} = -\frac{(x_2-x_1)}{y_2-y_1}\left\{x - \left(\frac{x_1+x_2}{2}\right)\right\} \quad \text{... (i)}$$

We know that

$$|\overline{AC}| = |\overline{BC}|$$

$$\Rightarrow \quad \sqrt{(x_1+g)^2 + (y_1+f)^2} = \sqrt{(x_2+g)^2 + (y_2+f)^2}$$

Squaring both sides

$$(x_1+g)^2 + (y_1+f)^2 = (x_2+g)^2 + (y_2+f)^2$$

$$\Rightarrow \quad x_1^2 + 2gx_1 + y_1^2 + 2fy_1 = x_2^2 + 2gx_2 + y_2^2 + 2fy_2$$

$$\Rightarrow \quad x_1^2 - x_2^2 + 2g(x_1 - x_2) = -(y_1^2 - y_2^2) - 2f(y_1 - y_2)$$

$$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2 + 2g) = -(y_1 - y_2)(y_1 + y_2 + 2f)$$

$$\Rightarrow \quad -\frac{(x_2-x_1)}{y_2-y_1} = \frac{(y_1+y_2+2f)}{x_1+x_2+2g}$$

Using this in equation (i), we get

$$2y - y_1 - y_2 = \frac{(y_1+y_2+2f)}{x_1+x_2+2g}(2x - x_1 - x_2)$$

$$\Rightarrow \quad (2y - y_1 - y_2)(x_1 + x_2 + 2g) - (y_1 + y_2 + 2f)(2x - x_1 - x_2) = 0 \quad \text{...(ii)}$$

This is equation of perpendicular bisector of chord $\overline{AB}$.

Now, we substitute centre $(-g, -f)$ in equation (ii), we get

$$(-2f - y_1 - y_2)(x_1 + x_2 + 2g) - (y_1 + y_2 + 2f)(-2g - x_1 - x_2) = 0$$

$$\Rightarrow \quad -(2f + y_1 + y_2)(x_1 + x_2 + 2g) + (y_1 + y_2 + 2f)(2g + x_1 + x_2) = 0$$

$$\Rightarrow \quad 0 = 0$$

$\because$ centre $C(-g, -f)$ satisfies equation (ii)

$\therefore$ Perpendicular bisector of the chord $\overline{AB}$ passes through the centre.

- **Line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.**

Let l be the line joining the centre $C(-g,-f)$ of the circle $x^2+y^2+2gx+2fy+c=0$ to the mid-point of chord. $\overline{AB}$ whose end points are $A(x_1,y_1)$ and $B(x_2,y_2)$ as shown in the figure 8.24.

Mid-point $D=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$\because \quad |\overline{AC}|=|\overline{BC}|$

$\therefore \quad \sqrt{(x_1+g)^2+(y_1+f)^2}=\sqrt{(x_2+g)^2+(y_2+f)^2}$

Squaring both sides

$$(x_1+g)^2+(y_1+f)^2=(x_2+g)^2+(y_2+f)^2$$

$$\Rightarrow \quad x_1^2+2gx_1+y_1^2+2fy_1=x_2^2+2gx_2+y_2^2+2fy_2$$

$$\Rightarrow \quad x_1^2-x_2^2+2g(x_1-x_2)=-(y_1^2-y_2^2)-2f(y_1-y_2)$$

$$\Rightarrow \quad (x_1-x_2)(x_1+x_2+2g)=-(y_1-y_2)(y_1+y_2+2f)$$

$$\Rightarrow \quad -\frac{(x_1+x_2+2g)}{y_1+y_2+2f}=\frac{y_2-y_1}{x_2-x_1} \qquad \ldots\text{(i)}$$

Now, slope of chord $\overline{AB}=\frac{y_2-y_1}{x_2-x_1}$

$$=-\frac{(x_1+x_2+2g)}{y_1+y_2+2f} \qquad \text{(Using equation (i))}$$

and slope of l = slope of $\overline{CD}$.

$$=\frac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g}$$

$$=\frac{y_1+y_2+2f}{x_1+x_2+2g}$$

Now, (slope of l) × (slope of $\overline{AB}$)

$$=\frac{y_1+y_2+2+f}{x_1+x_2+2g}\times\left\{-\frac{(x_1+x_2+2g)}{(y_1+y_2+2f)}\right\}$$

$$=-1$$

$\because$ product of slopes of l and chord $\overline{AB}$ is -1.

$\therefore$ The line l through the centre of circle and the mid-point of chord is perpendicular to the chord.

- **Congruent chords of a circle are equidistant from its centre and its converse**

Let $\overline{AB}$ and $\overline{CD}$ be two congruent chords of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $O(-g, -f)$ whereas the end of chords are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$ as shown in the figure 8.25.

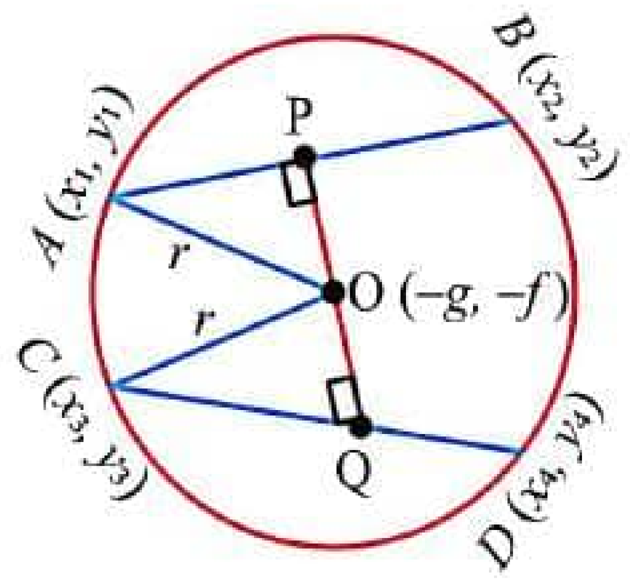


Fig. 8.25

Let $\overline{OP}$ is perpendicular distance from centre to chord $\overline{AB}$ then P is mid-point of chord $\overline{AB}$ according to property 1.

Also let $\overline{OQ}$ is perpendicular distance of centre to chord $\overline{CD}$ then Q is mid-point of chord $\overline{CD}$ according to property 1.

According to the condition

$$|\overline{AB}| = |\overline{CD}|$$

i.e., $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$

Squaring both sides

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2 \qquad \ldots(i)$$

In right angled ΔAOP

$$|\overline{OP}|^2 = |\overline{AO}|^2 - |\overline{AP}|^2$$

$$= r^2 - \left(\frac{1}{2}|\overline{AB}|\right)^2 \qquad (\because \ |\overline{OA}| = r)$$

$$= r^2 - \frac{1}{4}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}$$

$$\Rightarrow \quad |\overline{OP}| = \frac{\sqrt{4r^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2}}{2} \qquad \ldots(ii)$$

In right angled ΔCOQ

$$|\overline{OQ}|^2 = r^2 - |\overline{CQ}|^2$$

$$\Rightarrow \quad |\overline{OQ}|^2 = r^2 - \left(\frac{1}{2}|\overline{CD}|\right)^2$$

$$\Rightarrow \quad |\overline{OQ}|^2 = r^2 - \frac{1}{4}\{(x_4 - x_3)^2 + (y_4 - y_3)^2\}$$

$$\Rightarrow \quad |\overline{OQ}|^2 = r^2 - \frac{1}{4}\{(x_2 - x_1)^2 + (y_2 - y_1)^2\} \text{ (using equation (i))}$$

$$\Rightarrow \quad |\overline{OQ}| = \frac{\sqrt{4r^2 - (x_2 - x_1)^2 + (y_2 - y_1)^2}}{2} \qquad \ldots(iii)$$

From (ii) and (iii), we get

$$|\overline{OP}| = |\overline{OQ}|$$

Hence chords are equidistant from the centre.

- **Converse of Theorem 4a**

If the perpendicular distances from the centre of a circle to its two chords are equal, then the chords are congruent.

Let $\overline{AB}$ and $\overline{CD}$ be two chords of a circle $x^2+y^2+2gx+2fy+c=0$ with centre $O(-g,-f)$ where the ends of chords are $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ and $D(x_4,y_4)$ as shown in the Fig. 8.26.

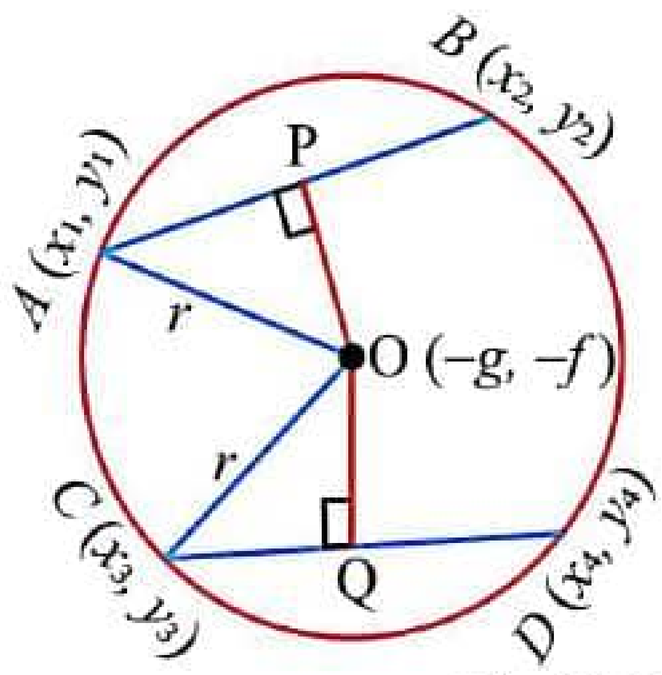


Fig. 8.26

Let $\overline{OP}$ is perpendicular distance of centre to the chord $\overline{AB}$, So P is the mid-point of chord $\overline{AB}$.

So, mid-point $P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Also let $\overline{OQ}$ is perpendicular distance of centre to the chord $\overline{CD}$, So Q is mid-point of chord $\overline{CD}$.

So, mid-point $Q = \left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

According to the condition

$$|\overline{OP}| = |\overline{OQ}|$$

i.e., $$\sqrt{\left(\frac{x_1+x_2}{2}+g\right)^2+\left(\frac{y_1+y_2}{2}+f\right)^2} = \sqrt{\left(\frac{x_3+x_3}{2}+g\right)^2+\left(\frac{y_3+y_3}{2}+f\right)^2}$$

Squaring both sides

$$\frac{(x_1+x_2+2g)^2}{4}+\frac{(y_1+y_2+f)^2}{4} = \frac{(x_3+x_4+2g)^2}{4}+\frac{(y_3+y_4+2f)^2}{4} \quad \ldots\text{(i)}$$

In right angled ΔAOP

$$|\overline{AP}|^2 = |\overline{AO}|^2 - |\overline{OP}|^2$$

$$\Rightarrow |\overline{AP}|^2 = r^2 - \frac{(x_1+x_2+2g)^2}{4} - \frac{(y_3+y_4+2f)^2}{4} \quad (\because \ |\overline{OA}| = r)$$

$$\Rightarrow \quad |\overline{AP}| = \frac{\sqrt{4r^2-(x_1+x_2+2g)^2-(y_3+y_4+2f)^2}}{2} \quad \ldots\text{(ii)}$$

In right angled ΔCOQ

$$|\overline{CQ}|^2 = |\overline{OC}|^2 - |\overline{OQ}|^2$$

$$\Rightarrow \quad |\overline{CQ}|^2 = r^2 - \left\{\frac{(x_3+x_4+2g)^2}{4}+\frac{(y_3+y_4+2f)^2}{4}\right\} \quad (\because \ r = |\overline{OC}|)$$

$$\Rightarrow \quad |\overline{CQ}|^2 = r^2 - \left\{\frac{(x_1+x_2+2g)^2}{4}+\frac{(y_1+y_2+2f)^2}{4}\right\} \quad \text{(using equation (i))}$$

$$\Rightarrow \quad |\overline{CQ}| = \frac{\sqrt{4r^2-(x_1+x_2+2g)^2-(y_1+y_2+2f)^2}}{2} \quad \ldots\text{(iii)}$$

From (ii) and (iii), we get

Now $\quad \tan(m\angle BAC) = \dfrac{m_4 - m_3}{1 + m_4 m_3}$

i.e., $\quad \tan \emptyset = \dfrac{\frac{-(b-y_1)}{x_1} - \frac{(b-y_1)}{x_1}}{1 - \frac{(b-y_1)}{x_1}\frac{(b-y_1)}{x_1}} \qquad$ (Let $m\angle BAC = \emptyset$)

$$= \frac{\frac{-2(b-y_1)}{x_1}}{\frac{x_1^2 - b^2 + 2by_1 - y_1^2}{x_1^2}}$$

$$= \frac{-2(b-y_1)x_1}{b^2 - y_1^2 - b^2 + 2by_1 - y_1^2} \qquad \text{(Using equation (ii))}$$

$$= \frac{-2(b-y_1)x_1}{-2y_1^2 + 2by_1}$$

$$= \frac{-2(b-y_1)x_1}{-2y_1(y_1 - b)}$$

$$= \frac{-2(y_1-b)x_1}{2y_1(y_1 - b)}$$

$$\tan \emptyset = -\frac{x_1}{y_1} \qquad \text{... (iii)}$$

We know that

$$\tan 2\emptyset = \frac{2\tan \emptyset}{1 - \tan^2 \emptyset}$$

$$= \frac{2\left(-\frac{x_1}{y_1}\right)}{1 - \frac{x_1^2}{y_1^2}}$$

$$= \frac{-2x_1y_1}{y_1^2 - x_1^2}$$

$$= \frac{2x_1y_1}{x_1^2 - y_1^2} \qquad \text{... (iv)}$$

From equation (i) and equation (iv)

$$\tan \emptyset = \tan 2\emptyset$$

$$\Rightarrow \quad \theta = 2\emptyset$$

Hence central angle of minor arc is double than the angle subtended by the corresponding major arc.

- **An angle in a semi-cricle is a right angle**

Let $P(x,y)$ be any point on the circle $x^2+y^2=r^2$ with radius r and centre at origin, whereas $A(r,0)$ and $B(-r,0)$ are two points of its diameter AB as shown in Fig. 8.28.

Now $\angle APB$ is the angle in semi-circle

Now,

Slope of $\overline{PB} = \frac{y}{x+r} = m_1$

and Slope of $\overline{AP} = \frac{y}{x-r} = m_2$

Product of slopes $= m_1 m_2$

$$= \left(\frac{y}{x+r}\right)\left(\frac{y}{x-r}\right)$$

$$= \frac{y^2}{x^2-r^2}$$

$$= \frac{y^2}{x^2-(x^2+y^2)} \quad (\because x^2+y^2=r^2)$$

$$= \frac{y^2}{-y^2} = -1$$

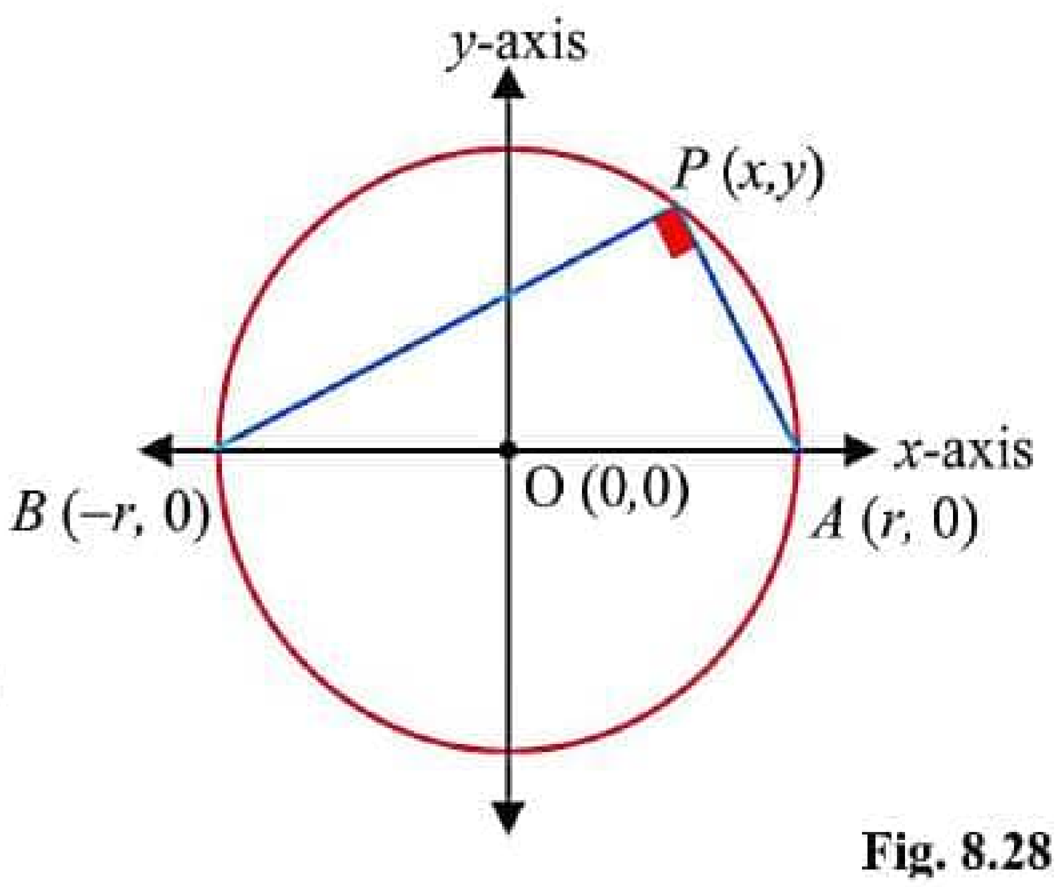


Fig. 8.28

$\because$ product of slopes $= -1$.

$\therefore$ $\overline{AP} \perp \overline{PB}$

Hence $\angle APB$ is right angle.

So, angle in a semi-circle is right angle.

- **The perpendicular at the outer end of radial segment is tangent to the circle**

Let l be the line perpendicular to the radial segment $\overline{CP}$ of circle $x^2+y^2+2gx+2fy+c=0$ at the outer end $P(x_1,y_1)$ whereas $C(-g,-f)$ is the centre of the circle as shown in Fig. 8.29.

Now,

Slope of radial segment $\overline{CP} = \frac{y_1+f}{x_1+g} = m$

So, slope of $l = -\frac{1}{m}$

$$= -\frac{(x_1+g)}{y_1+f} \quad \text{... (i)}$$

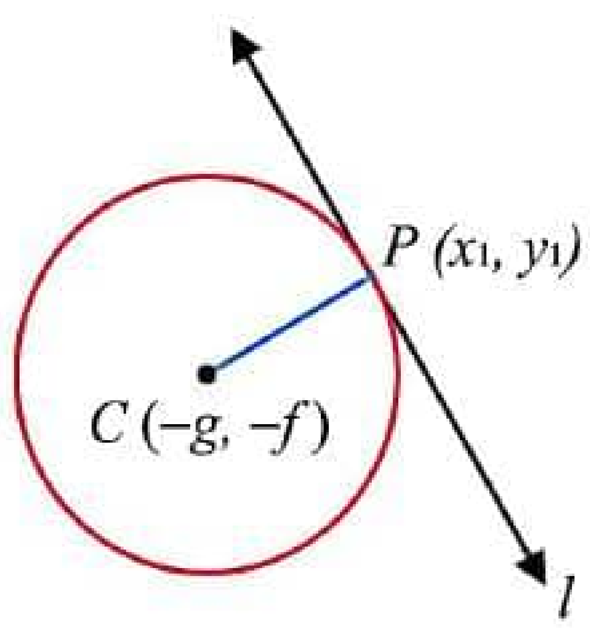


Fig. 8.29

We have

$$x^2+y^2+2gx+2fy+c=0$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (2y + 2f)\frac{dy}{dx} = -2x - 2g$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2(x+g)}{2(y+f)} = \frac{-(x+g)}{y+f}$$

Now,

Slope of tangent to the circle at $(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$= -\frac{(x_1 + g)}{y_1 + f} \quad \text{... (ii)}$$

From equation (i) and equation (ii)

Slope of l = slope of tangent at (x_1, y_1)

So, line l is tangent to the circle at (x_1, y_1).

Hence the perpendicular at the outer end of a radial segment is tangent to the circle.

- **The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.**

Let t be the tangent to the circle at any point $P(x, y)$ of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $C(-g, -f)$ whereas $\overline{CP}$ is the radial segment of the circle at the point P as shown in Fig. 8.30.

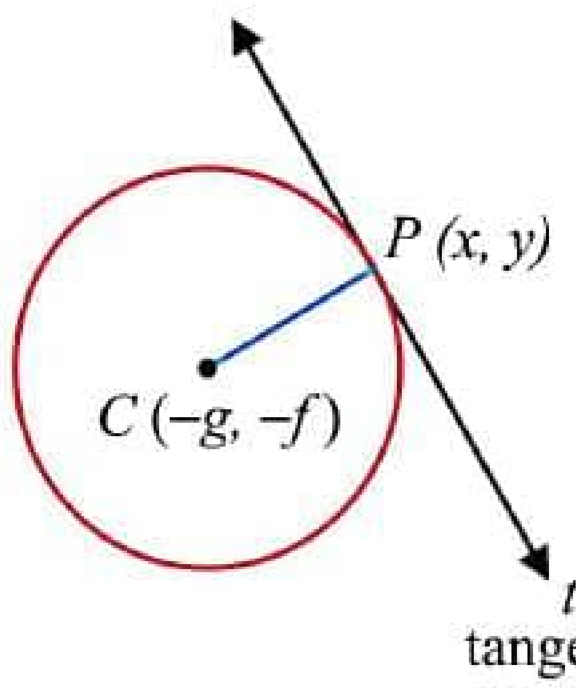


Fig. 8.30

Now,

Slope of radial segment $\overline{CP} = \frac{y+f}{x+g} = m_1$

We have the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2(y + f)\frac{dy}{dx} = -2(x + g)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x+g}{y+f}\right)$$

i.e., Slope of tangent to the circle at any point of the circle

$$= -\frac{(x + g)}{y + f} = m_2$$

Here $\quad m_1 m_2 = \left(\frac{y+f}{x+g}\right)\left[-\left(\frac{x+g}{y+f}\right)\right]$

$$= -1$$

∵ $m_1m_2 = -1$

∴ Tangent to the circle is perpendicular to the radial segment at the point of contact.

Exercise 8.4

Prove the following analytically

1. The tangents drawn at the ends of a diameter of a circle are parallel.
2. A normal to a circle passes through the centre of circle.
3. The mid-point of hypotenuse of a right triangle is the centre of the circle circumscribing the triangle.
4. Measure of the central angle of a major arc is double the measure of the inscribed angle of corresponding minor arc.
5. The parallelogram circumscribing a circle is a rhombus.

Review Exercise 8

1. Tick the correct option.

(i) If a plane cuts one nappe of a right circular cone perpendicularly then conic is -----

(a) parabola (b) circle (c) ellipse (d) hyperbola

(ii) The centre of circle with equation $(x+3)^2 + (y-5)^2 = 36$ is --------------

(a) $(3,-5)$ (b) $(-3,-5)$ (c) $(-3,5)$ (d) $(3,5)$

(iii) The centre of circle with equation $x^2 + y^2 + 10x - 8y + 1 = 0$ is --------------

(a) $(-5,\ 8)$ (b) $(-10,\ 8)$ (c) $(5,-4)$ (d) $(-5,\ 4)$

(iv) The radius of circle with equation $x^2 + y^2 + 4x + 6y + 1 = 0$ is --------------

(a) $\sqrt{13}$ (b) $\sqrt{12}$ (c) $\sqrt{10}$ (d) $\sqrt{71}$

(v) Which of the following is a degenerate conic

(a) circle (b) ellipse (c) line (d) parabola

(vi) The equation of circle with centre at origin and diameter of 10 units is ---------

(a) $x^2 + y^2 = 100$ (b) $x^2 + y^2 + 100 = 0$

(c) $x^2 + y^2 = 50$ (d) $x^2 + y^2 = 25$

(vii) For what value of k the radius of circle $x^2 + y^2 + 6x - 4y + k = 0$ is 5

(a) 11 (b) -12 (c) 10 (d) 12

(viii) The centre of the circle $x^2 + y^2 + 6x + 8 = 0$ is:

(a) on x-axis (b) on y-axis (c) in 1st quadrant (d) at origin

(ix) The circle $x^2 + y^2 + 6x + 10y + 9 = 0$

(a) touches x-axis (b) touches y-axis

(c) passes through origin (d) cuts x-axis

(x) The circle $x^2 + y^2 + 20x - 8y + 16 = 0$

(a) touches x-axis (b) touches y-axis

(c) passes through origin (d) cuts y-axis

(xi) The line $y = 2x + c$ will be tangent to $x^2 + y^2 = 25$ if

(a) $c^2 = 25$ (b) $c^2 = 625$ (c) $c^2 = 50$ (d) $c^2 = 125$

(xii) For what value of k, the line $y = 2x + 3$ is tangent to $x^2 + y^2 = k^2$

(a) $\pm\frac{5}{\sqrt{3}}$ (b) $\pm\frac{3}{\sqrt{5}}$ (c) $\pm\sqrt{\frac{3}{5}}$ (d) $\pm\frac{\sqrt{5}}{3}$

(xiii) For what value of k, the line $2x + 3y + k = 0$ is normal to the circle $x^2 + y^2 + 2x + 9 = 0$

(a) -2 (b) 2 (c) 3 (d) -3

(xiv) Equation of tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$ is:

(a) $3x + 4y = 0$ (b) $4x + 3y = 25$

(c) $3x + 4y = 25$ (d) $3x + 4y = 5$

(xv) Equation of normal to the circle $x^2 + y^2 = 36$ at $(2, 4\sqrt{2})$ is:

(a) $2x + 4\sqrt{2}y = 0$ (b) $4\sqrt{2}x + 2y = 0$

(c) $2x - 4\sqrt{2}y = 0$ (d) $4\sqrt{2}x - 2y = 0$

(xvi) The equation of tangent to $x^2 + y^2 = 100$ is ________ if slope of tangent is $\sqrt{15}$

(a) $y = \sqrt{15}x \pm 40$ (b) $y = -\sqrt{15x} \pm 40$

(c) $y = \sqrt{15x} \pm 40y$ (d) $y = -\sqrt{15x} \pm 40y$

(xvii) The length of tangent to the circle $x^2 + y^2 + 2y - 1 = 0$ from $(5, 2)$ is:

(a) $\sqrt{24}$ units (b) $\sqrt{33}$ units (c) $\sqrt{32}$ units (d) $\sqrt{31}$ units

(xviii) Congruent chords of a circle are equidistant from its

(a) diameter (b) centre (c) arc (d) segment

(xix) Angle in a semi-circle is --------------

(a) acute angle (b) obtuse angle (c) right angle (d) straight angle

(xx) The point $(3, 3)$ is ____________ the circle $x^2 + y^2 = 64$

(a) outside (b) inside

(c) on (d) cannot be determined

2. Find the equation of circle passing through $(2, 3), (4, 6)$ and

   (i) centre on x-axis (ii) centre on y-axis

3. $y = \sqrt{3}x + 10$ is the equation of tangent to the circle with centre at origin. Find the equation of normal to the circle at the point of tangent.

4. Find the condition of tangency, secancy and normality of line $x + y + k = 0$ to the circle $x^2 + y^2 + 2x - 3 = 0$.

Parabola, Ellipse and Hyperbola

Unit 9

Introduction

As a matter of fact, parabola, ellipse and hyperbola are the types of conic. In previous chapter, the Greek concept of conic was discussed in detail whereas the analytic concept was given in short.

In this chapter we will study conics analytically. Recall that in analytic geometry, a conic is the locus of a moving point or the set of all points whose distance from a fixed point (in the plane) bears a constant ratio to its distance from a fixed line in the same plane.

The fixed point, the fixed line and the constant ratio are called focus, directrix and eccentricity of the conic respectively, whereas eccentricity is denoted by e.

The line through the focus and perpendicular to the directrix is called the axis of the conic. The distance of a point on the conic from its focus is called the focal distance. The chord through the focus of a conic is called focal chord of the conic and the focal chord which is perpendicular to its axis is called the latus rectum.

Different conics are identified on the basis of the value of eccentricity as mentioned below.

If $e = 1$, the conic is called a parabola.

If $e < 1$, the conic is called an ellipse.

If $e > 1$, the conic is called a hyperbola.

whereas

$$e = \frac{\text{The distance from the focus to any point on the conic}}{\text{The distance from the directrix to that point on the conic}}$$

9.1 Parabola

9.1.1 Define parabola and its elements (i.e., focus, directrix, eccentricity, vertex, focal chord and latus rectum).

Parabola and its elements:

A parabola is the set of all points in the plane which are equidistant from a fixed line and a fixed point not on the line.

The fixed line is called the directrix of the parabola and the fixed point is called its focus.

The straight line through the focus and perpendicular to the directrix is called the axis of the parabola. The point where the parabola meets its axis is called the vertex of the parabola. A chord which passes through the focus is called focal chord and the focal chord which is perpendicular to the axis of parabola is called its latus rectum. Focus is the mid-point of latus rectum. In figure 9.1, P, Q and R are three points on the parabola whose focus is F and the vertex is V. The focal chord QR is its latus rectum. By definition of eccentricity,

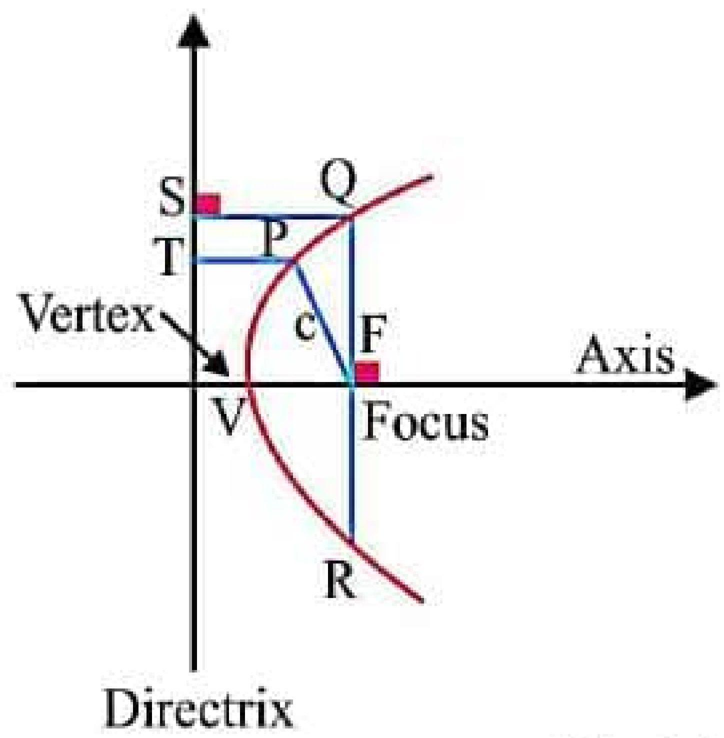


Fig 9.1

$$e = \frac{|\overline{PF}|}{|\overline{PT}|} \quad \text{or} \quad e = \frac{|\overline{QF}|}{|\overline{QS}|}$$

$$= 1 \quad \text{or} \quad e = 1 \qquad (\because \ |\overline{PF}| = |\overline{PT}| \text{ and } |\overline{QF}| = |\overline{QS}|)$$

So, the eccentricity of parabola is 1.

Note: The axis of parabola is also called the axis of symmetry of the parabola as the parabola is symmetric about its axis.

9.2 General Form of Equation of a Parabola

General form of equation of parabola means the equation which can be used to find parabola for any focus and any directrix.

9.2.1 Derive the general form of an equation of a parabola

Consider a parabola whose focus is $F(h,k)$ and equation of its directrix is $lx + my + n = 0$.

Let $P(x,y)$ be any point on the parabola. By definition of parabola

$$|\overline{PF}| = |\overline{PT}|$$

i.e., $$\sqrt{(x-h)^2 + (y-k)^2} = \left|\frac{lx+my+n}{\sqrt{l^2+m^2}}\right|$$

$\Rightarrow \quad (l^2 + m^2)[(x-h)^2 + (y-k)^2] = (lx + my + n)^2$

$\Rightarrow \quad l^2x^2 - 2hl^2x + l^2h^2 + m^2x^2 - 2hm^2x + h^2m^2 + l^2y^2 - 2kl^2y + k^2l^2$
$+m^2y^2 - 2km^2y + k^2m^2 = l^2x^2 + m^2y^2 + n^2 + 2lmxy + 2mny + 2lnx$

$\Rightarrow \quad m^2x^2 - 2lmxy + l^2y^2 - 2hl^2x - 2hm^2x - 2lnx - 2kl^2y - 2km^2y$
$-2mny + l^2h^2 + h^2m^2 + k^2l^2 + k^2m^2 - n^2 = 0$

$\Rightarrow \quad (mx - ly)^2 - 2(hl^2 + hm^2 + ln)x - 2(kl^2 + km^2 + mn)y + l^2h^2 + h^2m^2$
$+k^2l^2 + k^2m^2 - n^2 = 0$

$\Rightarrow \quad (mx - ly)^2 + 2gx + 2fy + c = 0 \qquad \ldots(i)$

where

$$g = -(hl + hm + ln)$$
$$f = -(kl + km + mn)$$
and $$c = l^2h^2 + h^2m^2 + k^2l^2 + k^2m^2 - n^2$$

Equation (i) is the general equation of parabola.

It is evident from the equation that second degree terms in the equation of parabola form a perfect square.

In case directrix is parallel to x-axis then $l = 0$. So, by adjusting values of g, f, c accordingly equation (i) is reduced to

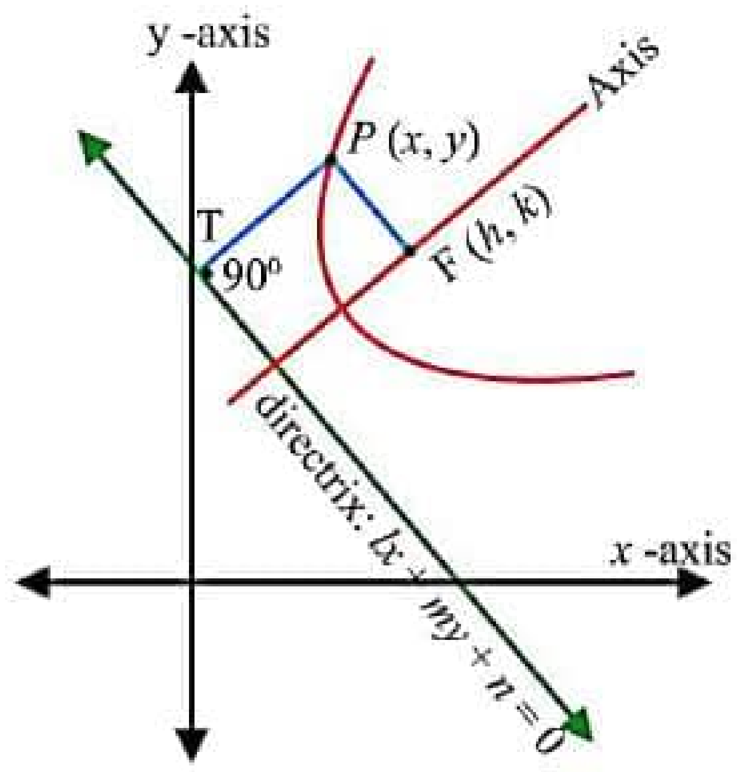


Fig 9.2

$$m^2x^2 + 2gx + 2fy + c = 0 \qquad \text{...(ii)}$$

In case directrix is parallel to y-axis then $m = 0$. So, by adjusting values of g, f, c accordingly equation (i) is reduced to

$$l^2y^2 + 2gx + 2fy + c = 0 \qquad \text{...(iii)}$$

Let $a = m^2$ and $b = l^2$

then

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(iv)}$$

represents the parabola whose directrix is parallel to either of axes if either $a = 0$ or $b = 0$.

Example: Find the equation of parabola whose focus is $F(3, 4)$ and directrix $l: 2x - 3 = 0$.

Solution: Let $P(x, y)$ be any point on the parabola.

According to the definition of parabola

$|\overline{PF}|$ = distance of P from l

i.e., $\sqrt{(x-3)^2 + (y-4)^2} = \left|\frac{2x-3}{2}\right|$

$\Rightarrow \quad 4\{(x-3)^2 + (y-4)^2\} = (2x-3)^2$

$\Rightarrow \quad 4(x^2 - 6x + 9 + y^2 - 8y + 16) = 4x^2 - 12x + 9$

$\Rightarrow \quad 4x^2 + 4y^2 - 24x - 32y + 100 = 4x^2 - 12x + 9$

$\Rightarrow \quad 4y^2 - 12x - 32y + 91 = 0$

This is the required equation of parabola.

9.3 Standard Form of Equation of Parabola

The four possible orientations of parabola such that vertex is at origin and the axis of parabola is along the x-axis or y-axis, are called the standard positions of a parabola and the resulting equations are called the standard equations of a parabola.

9.3.1 Derive the standard equations of parabola, sketch their graphs and find their elements:

(a) Standard equations of parabola

- **Standard equation of parabola when axis of parabola is along x-axis and vertex is at origin.**

Consider a parabola whose vertex is at origin and axis of symmetry is along x-axis as shown in the figure 9.3.

Let $F(a,0)$ be the focus on x-axis then the equation of directrix l will be $x=-a$ or $x+a=0$.

Let $P(x,y)$ be any point on the parabola, where $a \neq 0$. (Fig. 9.3)

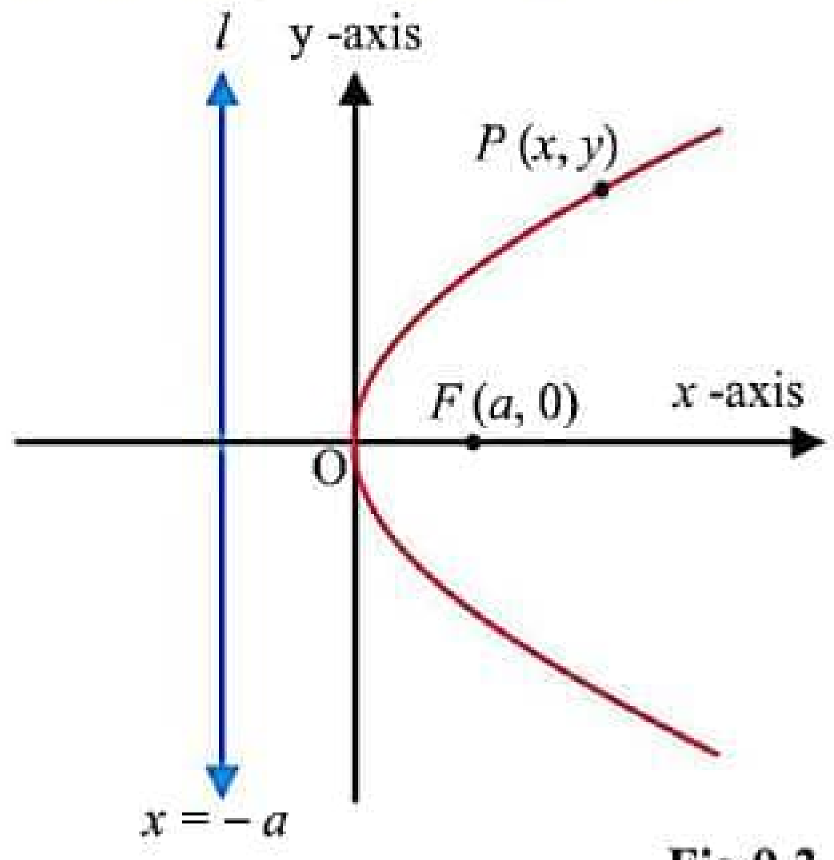


Fig 9.3

According to the definition of parabola

$|\overline{PF}|$ = distance of P from l

i.e., $\sqrt{(x-a)^2+y^2}=|x+a|$

Squaring both sides

$$(x-a)^2+y^2=(x+a)^2$$

$$\Rightarrow \quad y^2=4ax$$

This is the required standard equation of parabola.

If $a>0$ then it is cup-right parabola.

If $a<0$ then it is cup-left parabola as shown in fig. 9.4.

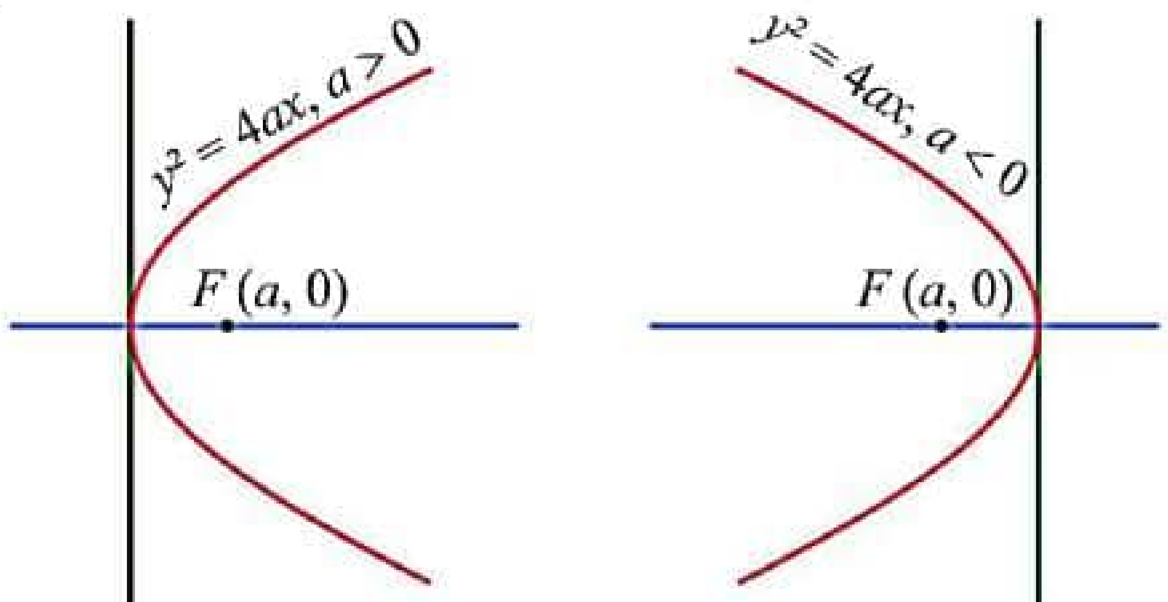


Fig 9.4

- **Standard equation of parabola when axis of parabola is along y-axis and vertex is at origin.**

By using definition of parabola, we can derive standard equation of parabola, when axis of parabola is along y-axis and vertex is at origin which is

$$x^2=4ay$$

where $F(0,a)$ is focus and equations of directrix is $y=-a$ as shown in fig. 9.5

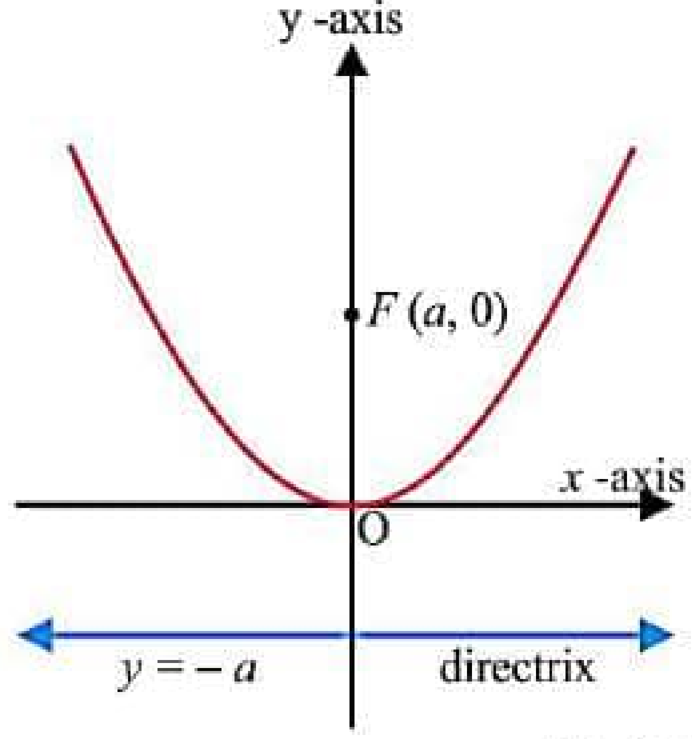


Fig 9.5

If $a>0$ then it is cup-up parabola.

If $a < 0$ then it is cup-down parabola as shown in fig. 9.6.

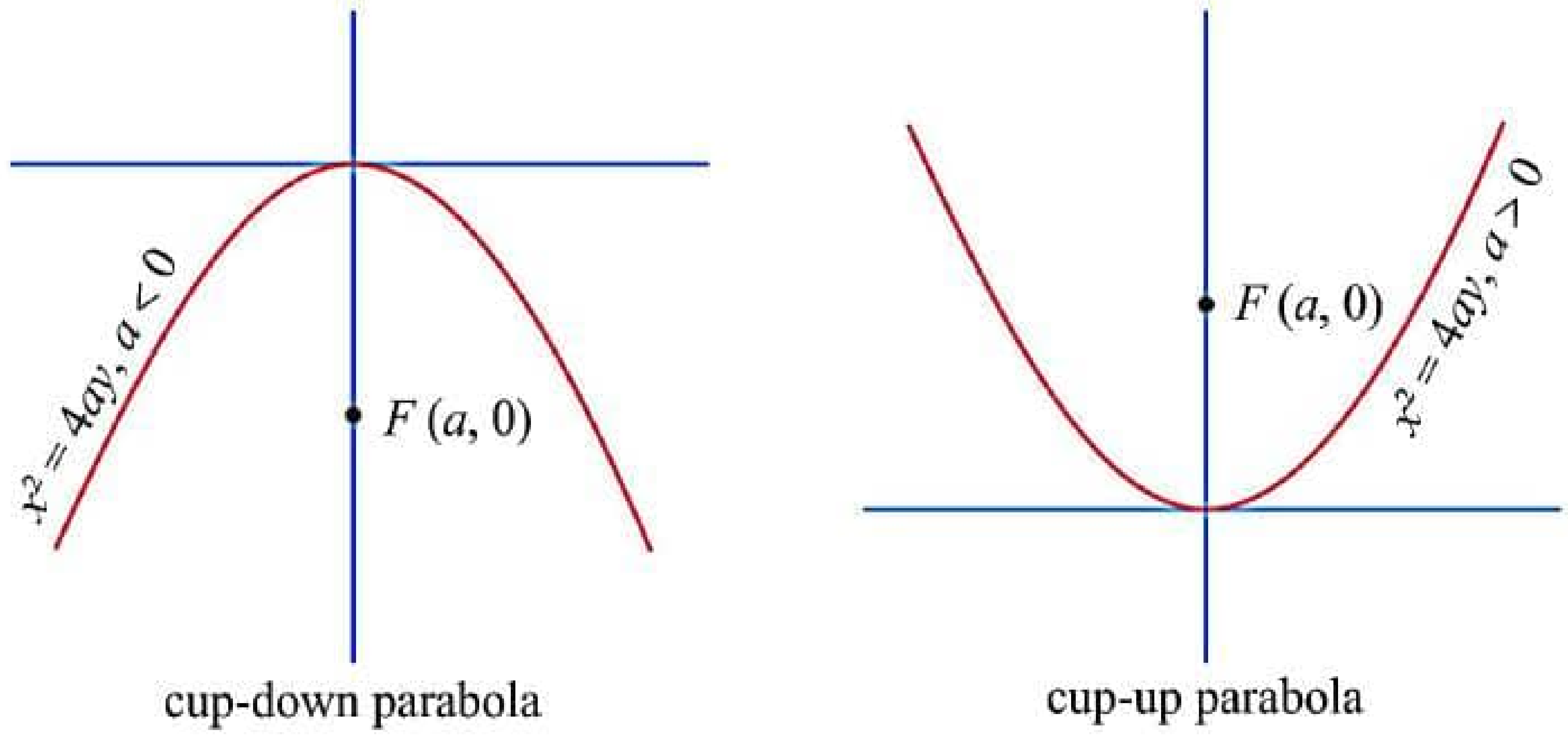


Fig. 9.6

Latus Rectum:

We know that the chord through the focus of a parabola and perpendicular to its axis is called the latus rectum of the parabola. In the Fig. 9.7 $\overline{AB}$ is latus rectum.

Here $\quad |\overline{AB}| = 2|\overline{AF}| = 2|\overline{AC}|$

or $\quad |\overline{AB}| = 2|\overline{EF}| = 2(2a) = 4a$

Thus, the length of the latus rectum is $|4a|$.

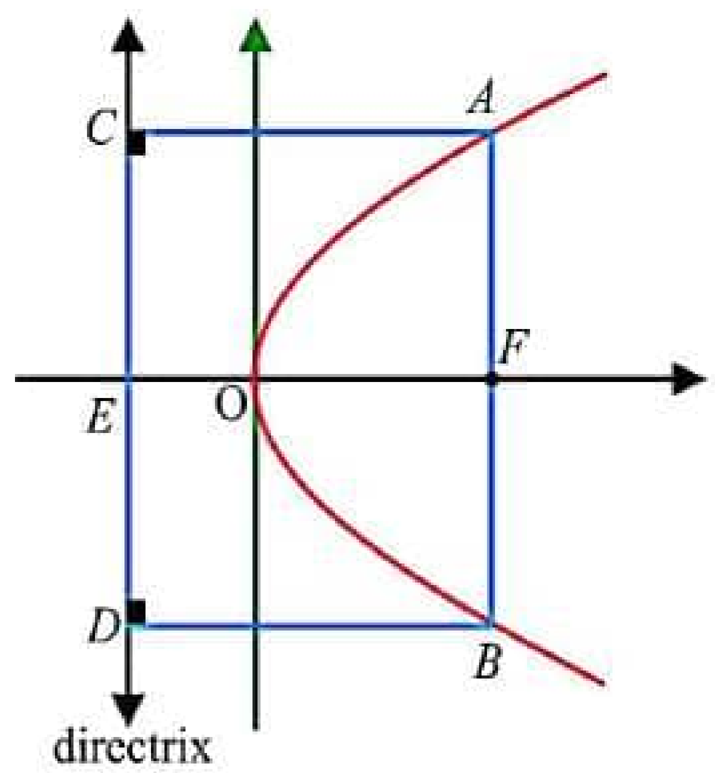


Fig 9.7

(b) Sketching the graph of parabolas from their standard equations

Graphs of parabolas from their standard equations can be sketched using the following steps.

1. Determine the axis of parabola from the given standard equations. If equation contains x^2-term, then its axis of symmetry is along y-axis. If equation contains y^2-term then its axis of symmetry is along x-axis.
2. Determine, in which way, the parabola opens. If parabola is along x-axis then it is cup-right and cup-left if $a > 0$ and $a < 0$ respectively. If parabola is along y-axis then it is cup-up and cup-down if $a > 0$ and $a < 0$ respectively.
3. Locate focus and draw the latus rectum of length $|4a|$.
4. Sketch parabola joining the ends of latus rectum with its vertex.

Example: Sketch the graphs of the following parabolas.

(i) $y^2 = 12x$ $\qquad$ (ii) $x^2 = -10y$

Solution:

(i) $y^2 = 12x$

Comparing with $y^2 = 4ax$

We get $4a = 12 \quad \Rightarrow \quad a = 3$

$\because$ equation has y^2 −term

$\therefore$ parabola is along x-axis and it is cup-right because $a > 0$.

Here, latus rectum $= |4a| = |4(3)| = 12$

Now, we draw latus rectum $\overline{AB}$ through focus $F(3, 0)$ and sketch the parabola as shown in figure 9.8.

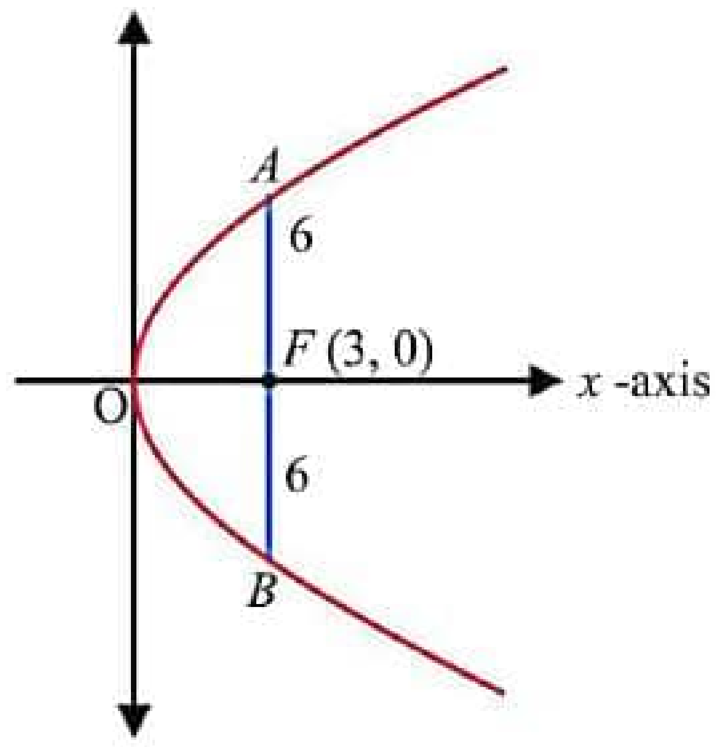


Fig 9.8

(ii) $x^2 = -10y$

Comparing with $x^2 = 4ay$

We get $4a = -10 \quad \Rightarrow \quad a = -\frac{5}{2}$

$\because$ equation has x^2 −term

$\therefore$ parabola is along y-axis and it is cup-down as $a < 0$.

Here, latus rectum $= |4a| = \left|4\left(-\frac{5}{2}\right)\right| = 10$

Now, we draw latus rectum $\overline{AB}$ through focus $F\left(0, -\frac{5}{2}\right)$ and sketch the parabola as shown in figure 9.9.

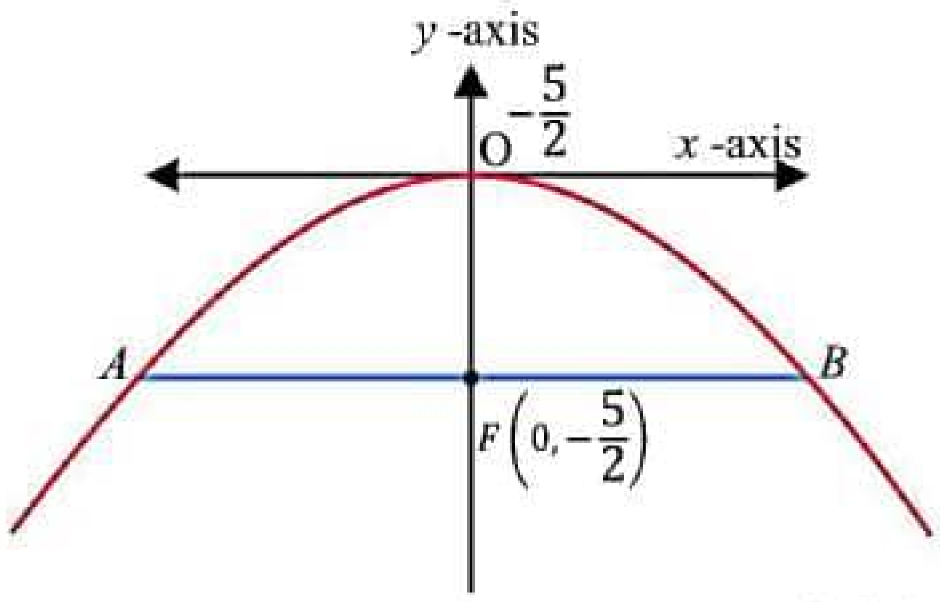


Fig 9.9

Standard forms of translated equations of parabola

The topic of translation and rotation of axes will be discussed in detail in section 9.12. At this stage we should know that if standard parabola is translated h units horizontally and k units vertically then its vertex will be (h, k) and the resulting equations will be

(1) $(y-k)^2 = 4a(x-h)$ in case axis of symmetry is parallel to x-axis as shown in the Fig. 9.10.

Here focus and directrix will be $(h+a, k)$ and $x = h - a$ respectively.

(2) $(x-h)^2 = 4a(y-k)$ in case axis of symmetry is parallel to y-axis as shown in the Fig. 9.11.

Here focus and directrix are $(h, k+a)$ and $y = k - a$ respectively.

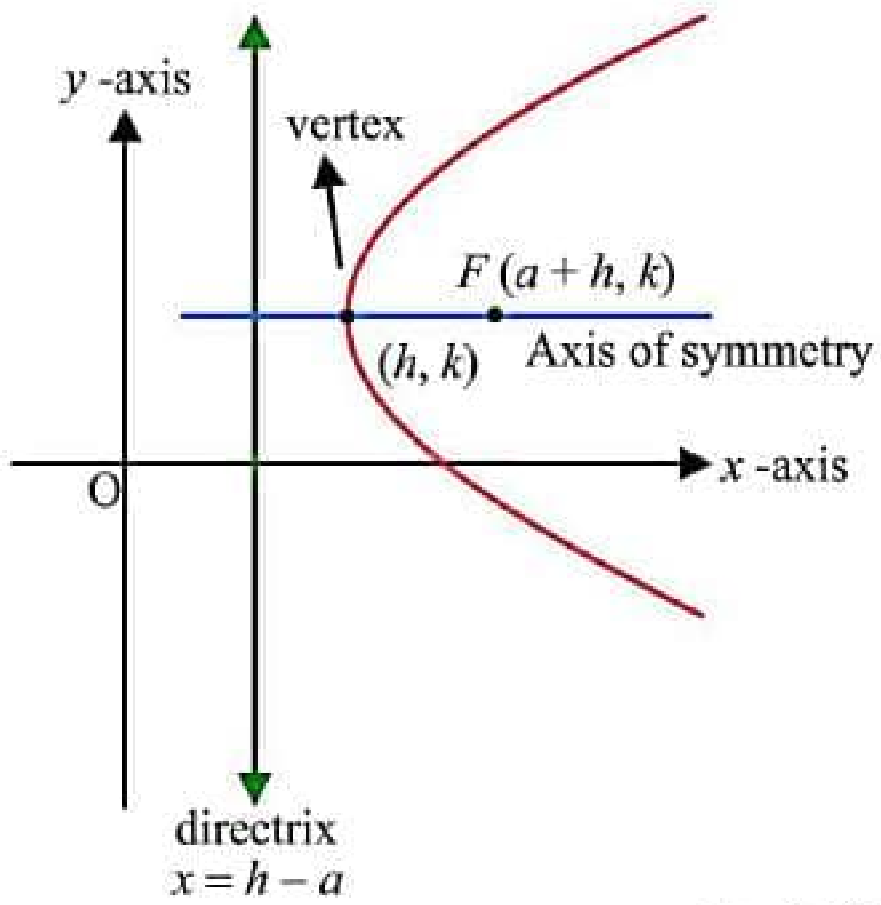


Fig 9.10

In order to find elements of parabola from its equation, we summarize the equations along with its elements as under:

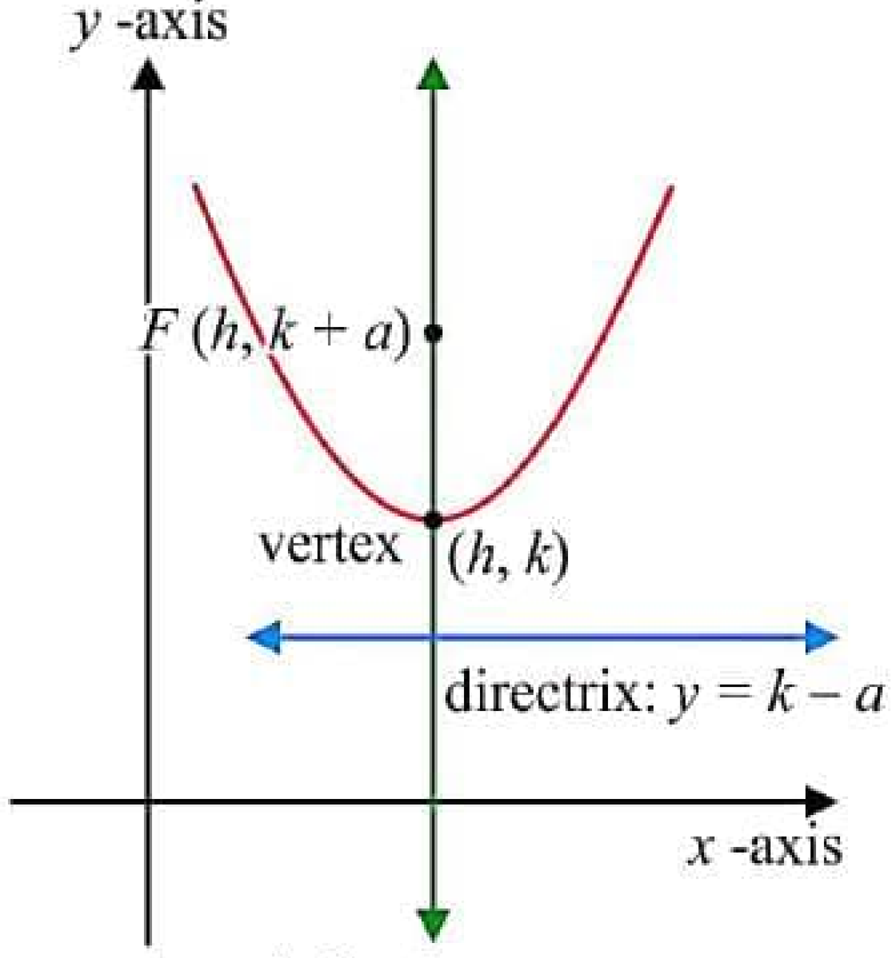


Fig 9.11

Equation of Parabola	Related information
(1) $y^2 = 4ax$	• Axis of symmetry is along x-axis with vertex at origin. Axis of symmetry: $y = 0$ • If $a > 0$ then it is cup-right. • If $a < 0$ then it is cup-left. • Focus is $(a, 0)$ • Latus rectum $= \|4a\|$ • Directrix is $x = -a$ • End points of latus rectum $= (a, \pm 2a)$
(2) $x^2 = 4ay$	• Axis of symmetry is along y-axis with vertex at origin. Axis of symmetry: $x = 0$. • If $a > 0$ then it is cup-up. • If $a < 0$ then it is cup-down. • Focus is $(0, a)$ • Latus rectum $= \|4a\|$ • Directrix is $y = -a$ • Ends points of latus rectum $= (\pm 2a, a)$
(3) $(y - k)^2 = 4a(x - h)$	• Axis of symmetry is parallel to x-axis with vertex $(h, k)y - k = 0$. • If $a > 0$ then it is cup-right. • If $a < 0$ then it is cup-left.

Equation of Parabola	Related information
	• Focus is $(h+a,k)$ • Latus rectum $= \lvert 4a \rvert$ • Directrix is: $x-h=-a$ • End points of latus rectum $=(h+a,k\pm a)$
(4) $(x-h)^2=4a(y-k)$	• Axis of symmetry is parallel to y-axis i.e., $x-h=0$ • If $a>0$ then it is cup-up. • If $a<0$ then it is cup-down. • Focus is $(h,k+a)$ • Latus rectum $= \lvert 4a \rvert$ • Directrix is: $y-k=-a$ • End points of latus rectum $=(h\pm 2a,k+a)$

(c) Finding elements of parabola

We find different elements of parabola with the help of the following examples.

Example 1. Find focus, latus rectum and equation of directrix of the parabola with equation $y^2=12x$. Also sketch its graph.

Solution: Given parabola is: $y^2=12x$

comparing with $y^2=4ax$

We get, $4a=12$

$\Rightarrow \quad a=3$

$\because$ Parabola is along x-axis with vertex $(0,0)$.

$\therefore$ Its focus $=(a,0)$

$=(3,0)$

Now, latus rectum $=|4a|$

$=|4(3)|$

$=|12|=12$

Equation of directrix will be

$x=-a$

i.e., $x=-3$

or $x+3=0$

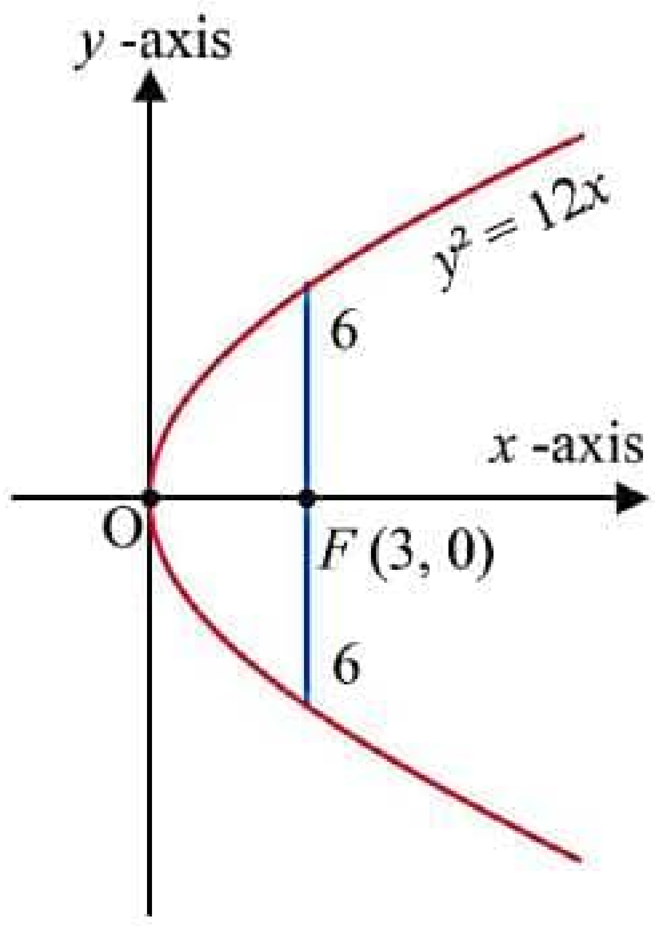


Fig 9.12

Graph of Parabola

Here,

Axis of symmetry is along x-axis with vertex at origin.

$\because \quad a>0$

$\therefore$ Its is cup-right parabola and latus rectum is 12 units.

This graph is shown in Fig. 9.12.

Example 2. Find vertex, focus, latus rectum, equation of axis and directrix of parabola $(x+2)^2 = -8(y-3)$. Also sketch its graph.

Solution: Given parabola is $(x+2)^2 = -8(y-3)$

comparing with $(x-h)^2 = 4a(y-k)$

We get, $h = -2, k = 3$ and $4a = -8$

$\Rightarrow \quad a = -2$

$\because$ Axis of symmetry is parallel to y-axis.

$\therefore$ Its focus $= (h, k+a)$

$= (-2, 3-2)$

$= (-2, 1)$

Its vertex $= (h, k) = (-2, 3)$

Now, latus rectum $= |4a|$

$= |-8| = 8$ units

Equation of axis will be

$x = h$

i.e., $x = -2$

or $x + 2 = 0$

Equation of directrix will be

$y - k = -a$

$\Rightarrow \quad y - 3 = 2$

$\Rightarrow \quad y - 5 = 0$

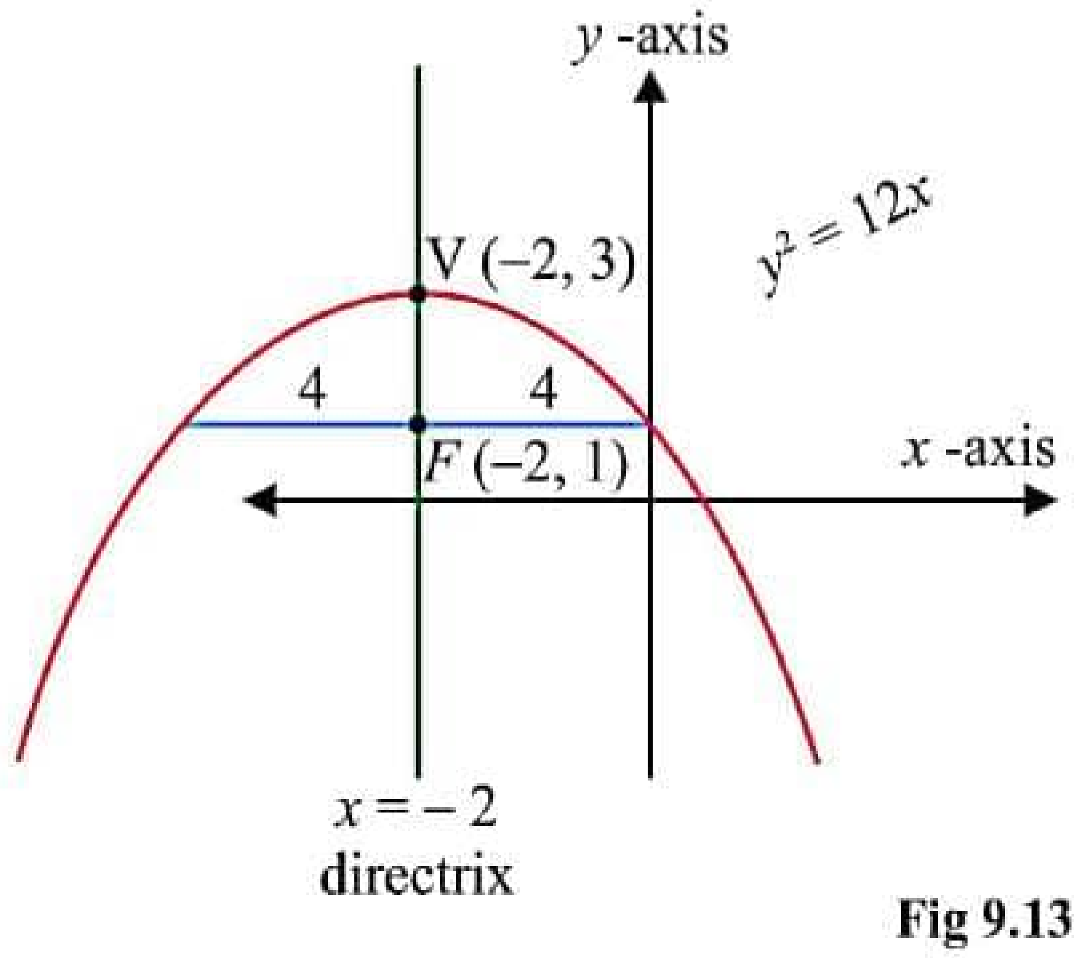


Fig 9.13

Graph of Parabola

Here, axis of symmetry is $x = -2$, which is parallel to y-axis. Vertex, focus and latus rectum are $(-2, 3), (-2, 1)$ and 8 respectively.

$\because \quad a < 0$

$\therefore$ Its is cup-down parabola. The graph is shown in Fig. 9.13.

9.3.2 Find the equation of a parabola with the following given elements:

- **focus and vertex,**
- **focus and directrix,**
- **vertex and directrix,**
- **vertex and points.**

(i) Equation of parabola when focus and vertex are given.

The method of finding equation of parabola when focus and vertex are given is explained with the help of the following examples.

Example 1. Find the equation of parabola when focus is $(5, 0)$ and vertex is $(0, 0)$.

Solution: Here focus $= (5, 0) = (a, 0)$

So $a = 5$

$\because$ Focus is on x-axis and vertex is at origin

$\therefore$ Its equation will be

$y^2 = 4ax$

i.e., $y^2 = 4(5)x$

or $y^2 = 20x$

Example 2. Find equation of parabola whose vertex is $(2, 3)$ and focus is $(2, 7)$.

Solution: Here, vertex $= (2, 3) = (h, k)$ and focus $= (2, 7) = (h, k + a)$

So, $k + a = 7$

$\Rightarrow a = 4$

According to the condition axis of symmetry is parallel to y-axis with vertex (h, k).

So, its equation will be

$$(x - h)^2 = 4a(y - k)$$

By using values of a, h, k

We get, $(x - 2)^2 = 4(4)(y - 3)$

$\Rightarrow x^2 - 4x + 4 = 16y - 48$

$\Rightarrow x^2 - 4x - 16y - 52 = 0$

(ii) Equation of parabola when focus and directrix are given

Method of finding equation of parabola when focus and directrix are given is explained with the following examples.

Example: Find the equation of parabola whose focus is $(2, 4)$ and equation of directrix is $x + 3 = 0$.

Solution:

Here, focus $= (2, 4)$ and directrix is $x + 3 = 0$.

Let $P(x, y)$ be any point of parabola.

So, $|\overline{PF}|$ = distance of P from directrix

i.e., $\sqrt{(x - 2)^2 + (y - 4)^2} = |x + 3|$

Squaring both sides

$$(x - 2)^2 + (y - 4)^2 = (x + 3)^2$$

$\Rightarrow x^2 - 4x + 4 + y^2 - 8y + 16 = x^2 + 6x + 9$

$\Rightarrow y^2 - 8y - 10x + 11 = 0$

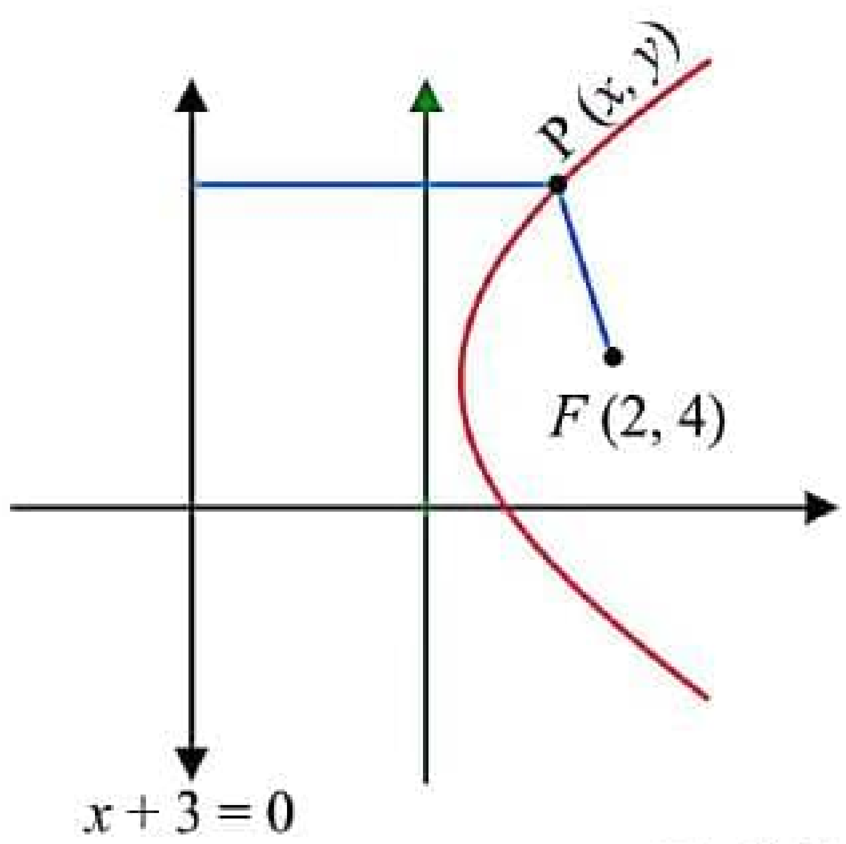


Fig 9.14

This is the required equation of parabola.

(iii) Equation of parabola when vertex and directrix are given

Method of finding equation of parabola when vertex and directrix are given is explained with the help of the following examples.

Example 1. Find the equation of parabola whose directrix is $x = 5$ and vertex is at origin.

Solution:

$\because$ directrix is parallel to y-axis and vertex is origin.

$\therefore$ axis of symmetry is along x-axis and its equation will be

$$y^2 = 4ax \quad \ldots\text{(i)}$$

with directrix $\quad x = -a \quad \ldots$(ii)

whereas given directrix is: $\quad x = 5 \quad \ldots$(iii)

comparing equations (ii) and (iii)

we get $\quad a = -5$

By using $a = -5$ in equation (i)

we get, $y^2 = -20x$

This is the required equations of parabola.

Example 2. Find the equation of parabola whose vertex is $(1, 2)$ and directrix is $y = 4$.

Solution:

$\because$ directrix is parallel to x-axis and vertex is not at origin.

$\therefore$ Axis of symmetry will be parallel to y-axis and its equation will be

$(x - h)^2 = 4a(y - k) \quad \ldots$(i)

with vertex $\quad (h, k)$ and directrix $y = k - a \quad \ldots$(ii)

Given directrix is $y = 4 \quad \ldots$(iii)

Here vertex $= (h, k) = (1, 2)$

comparing equation (ii) and (iii)

we get, $k - a = 4$

i.e., $\quad 2 - a = 4$

$\Rightarrow \quad a = -2$

Using $\;a = -2, h = 1$ and $k = 2$ in equation (i)

we get, $(x - 1)^2 = -8(y - 2)$

This is the required equation of parabola.

(iv) Equation of parabola when vertex and point are given:

The method of finding equation of parabola when vertex and point are given is explained by the following example.

Example: Find the equation of parabola whose vertex is $(0, 0)$ and passes through $(1, 2)$.

Solution:

$\because$ Vertex is at origin.

$\therefore$ Axis of symmetry may be along x-axis or y-axis.

Case I: When axis of symmetry is along x-axis

Let $\quad y^2 = 4ax \quad \ldots$(i)

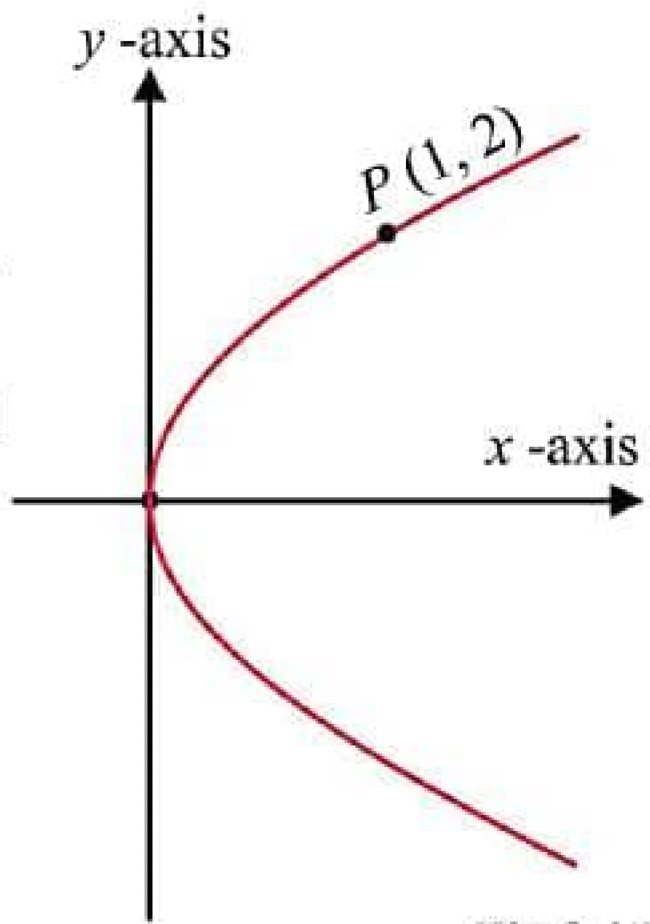


Fig 9.15

be the equation of parabola

$\because$ point $P(1,2)$ lies on the parabola

$\therefore$ equation (i) becomes

$$4 = 4a \qquad \Rightarrow \qquad a = 1$$

By using $a = 1$ in equation (i)

we get,

$y^2 = 4x$, which is the required equation of parabola.

Case II: When axis of symmetry is along y-axis

Let $x^2 = 4ay$ ...(i)

be the equation of parabola

$\because$ point $P(1,2)$ lies on the parabola

$\therefore$ equation (i) becomes

$$1 = 8a \qquad \Rightarrow \qquad a = \frac{1}{8}$$

By using $a = \frac{1}{8}$ in equation (i)

we get

$$x^2 = \frac{1}{2}y$$

This is the required equation.

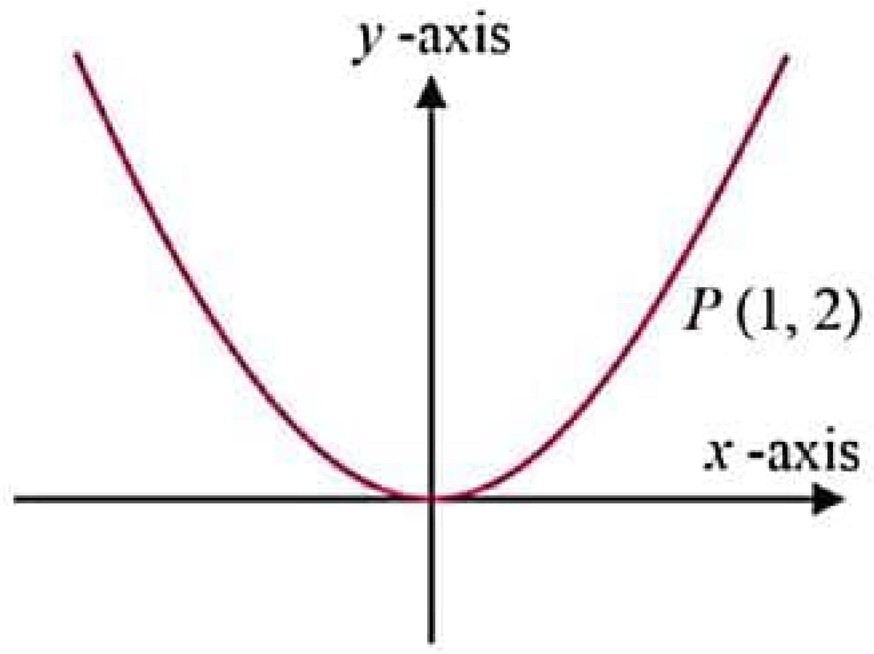


Fig 9.16

Exercise 9.1

1. Draw the following parabolas:
   (i) $y^2 = 10x$ (ii) $x^2 = -12y$
   (iii) $y^2 - x - 2y - 1 = 0$ (iv) $x^2 - 6x - 2y + 5 = 0$
2. Determine vertex, focus, latus rectum and equation of directrix of the following. Also find the equation of the axis of symmetry.
   (i) $y^2 = -8x$ (ii) $x^2 = -16y$
   (iii) $(y+3)^2 = 12(x-2)$ (iv) $(x+5)^2 = 8(y-3)$
   (v) $x^2 + 4x - y + 5 = 0$ (vi) $y^2 - 6y + 8x - 23 = 0$
3. Find the equation of parabola whose focus is $F(1,-2)$ and directrix is $3x - 5 = 0$.
4. Find the equation of the parabola whose focus is $(3,4)$ and the directrix is the line $x + y - 1 = 0$.
5. Find the equation of the parabolas whose focus and vertex are as under:
   (i) Vertex $(0,0)$; focus $(5,0)$ (ii) Vertex $(0,0)$; focus $(0,-2)$
   (iii) Vertex $(1,-3)$; focus $(1,2)$ (iv) Vertex $(2,4)$; focus $(3,4)$

6. Find the equation of parabola whose focus and directrix are given:
   (i) focus $(3,0)$ and directrix $x-5=0$
   (ii) focus $(0,4)$ and directrix $y+6=0$
   (iii) focus $(-4,3)$ and directrix $y=6$
7. Find equation of the parabola whose vertex and directrix are as under:
   (i) vertex $(0,0)$; directrix $x=-6$
   (ii) vertex $(0,0)$; directrix $y=5$
   (iii) vertex $(3,4)$; directrix $x=5$
8. Find the equation of parabola whose vertex and point are given:
   (i) vertex $(0,0)$; point $(3,4)$
   (ii) vertex $(5,0)$; point $(4,6)$
9. Find the standard equation of parabola whose latus rectum and vertex are the diameter and centre of the circle respectively x^2+y^2-4x and$-8y-5=0$.
10. Find the equation of circle and its circle is at the focus, whose diameter is the latus rectum of the parabola $x^2=12y$ and its centre is at the focus.
11. For what point of the parabola $y^2=10x$, the abscissa is equal to three times its ordinate.

9.4 Equation of Tangent and Normal of Parabola

In this section we will study about tangent and normal to a parabola along with their equations and conditions.

9.4.1 Recognize tangent and normal to a parabola

We know that a line which touches a parabola at a single point is called tangent and the line perpendicular to the point of tangency is called normal. In the figure 9.17, the line l is tangent to the parabola at point P whereas the line m is normal.

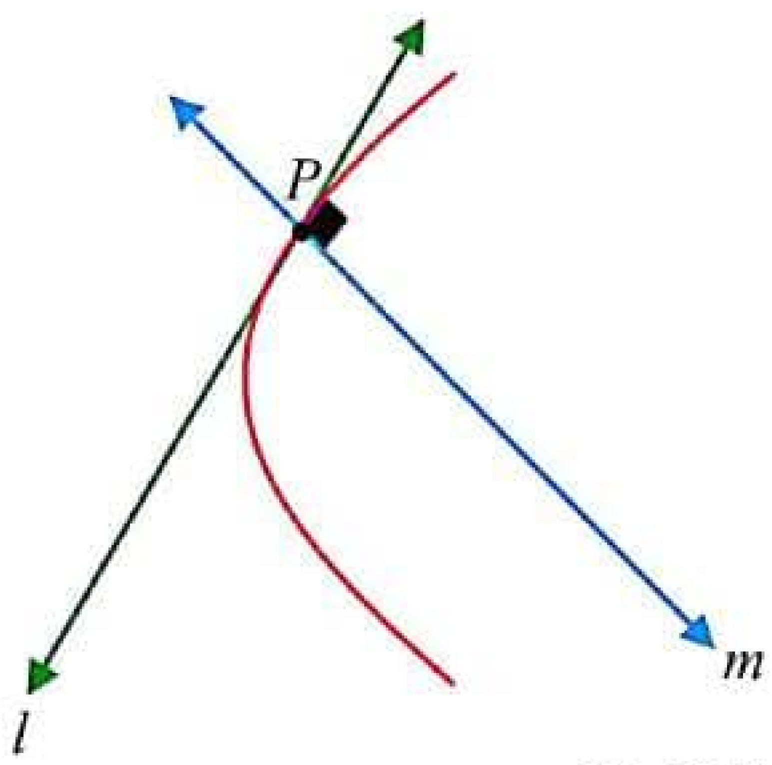


Fig 9.17

9.4.2 Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form

Consider a line

$l: y=mx+c$ ...(i)

and parabola $y^2=4ax$ ...(ii)

By using $y = mx + c$ in equation (i),

we get $(mx + c)^2 = 4ax$

$\Rightarrow \quad m^2x^2 + 2cmx + c^2 = 4ax$

$\Rightarrow \quad m^2x^2 + 2(cm - 2a)x + c^2 = 0 \quad \ldots$(iii)

Given line will be tangent to the parabola

if $\Delta = 0$ (where Δ is discriminant of equation (iii))

i.e., $\quad 4(cm - 2a)^2 - 4c^2m^2 = 0$

$\Rightarrow \quad c^2m^2 - 4acm + 4a^2 - c^2m^2 = 0$

$\Rightarrow \quad 4a^2 = 4acm$

$\Rightarrow \quad a = cm$

$\Rightarrow \quad c = \frac{a}{m}$

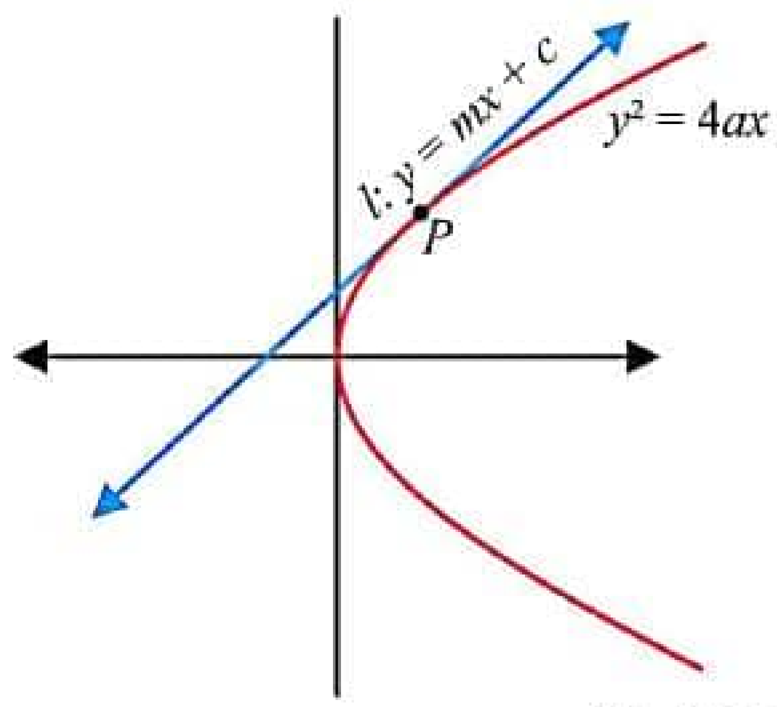


Fig 9.18

This is the condition of tangency of $y = mx + c$ to the parabola $y^2 = 4ax$.

By using $c = \frac{a}{m}$ in equation (i) (since $a \neq 0$)

we get,

$$y = mx + \frac{a}{m}$$

This is the equation of tangent to parabola $y^2 = 4ax$ in slope form.

From equation (iii), by quadratic formula

we have $x = -\frac{2(cm-2a)}{2m^2}$

$= -\frac{(a - 2a)}{m^2} \left(\because \ c = \frac{a}{m}\right)$

$= \frac{a}{m^2}$

By using $x = \frac{a}{m^2}$ in equation (i)

we get,

$y = m\left(\frac{a}{m^2}\right) + c$

$\Rightarrow \quad y = \frac{2a}{m} \left(\because \ c = \frac{a}{m}\right)$

So, the point of tangency is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Example: Find the condition when the line $2x + 3y = p$ is tangent to the parabola $y^2 = 12x$. Also find equation of tangent and point of tangency.

Solution: We have parabola $y^2 = 12x$ in which $a = 3$ and the line $l: 2x + 3y = p$

i.e., $\quad y = -\frac{2}{3}x + \frac{p}{3}$

Comparing with $y = mx + c$

we get, $m = -\frac{2}{3}$ and $c = \frac{p}{3}$

Now condition of tangency is:

$$c = \frac{a}{m}$$

i.e., $$\frac{p}{3} = \frac{3}{-\frac{2}{3}}$$

$$\Rightarrow \quad p = -\frac{27}{2}$$

Equation of tangent will be $y = mx + \frac{a}{m}$

i.e., $$y = -\frac{2}{3}x + \frac{3}{-\frac{2}{3}}$$

$$\Rightarrow \quad y = -\frac{2x}{3} - \frac{9}{2}$$

This is the required equation of tangent.

Now the point of tangency $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$$= \left(\frac{3}{\frac{4}{9}}, \frac{6}{-\frac{2}{3}}\right)$$

$$= \left(\frac{27}{4}, -9\right)$$

9.4.3 Find the equation of a tangent and a normal to a parabola at a point

Let $P(x_1, y_1)$ be a point of parabola $y^2 = 4ax$,

So $y_1^2 = 4ax_1$

Differentiating w.r.t x

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

Now slope of tangent at $P(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

i.e., $$m = \frac{2a}{y_1}$$

By point slope form, the equation of tangent will be

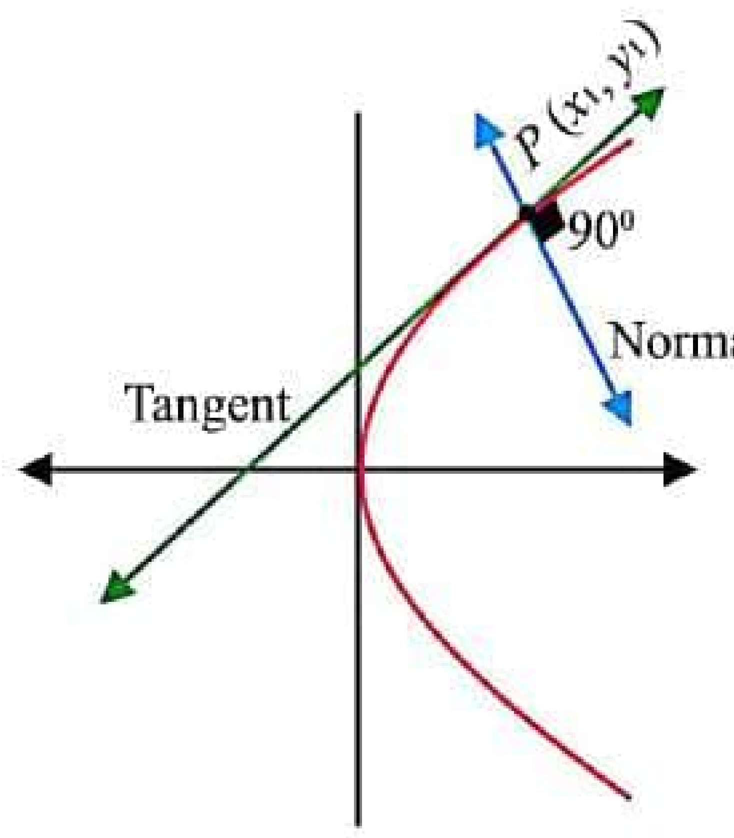


Fig 9.19

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = \frac{2a}{y_1}(x - x_1)$

$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$

$\Rightarrow \quad yy_1 - 4ax_1 = 2ax - 2ax_1 \qquad (\because y_1^2 = 4ax_1)$

$\Rightarrow \quad yy_1 = 2a(x + x_1)$

This is the equation of tangent to $y^2 = 4ax$ at (x_1, y_1).

$\because$ Normal is perpendicular to the tangent at point of contact $P(x_1, y_1)$

$\therefore$ slope of normal $= -\frac{1}{m}$

$$= -\frac{y_1}{2a}$$

Now equation of normal, by point slope form is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$\Rightarrow \quad y_1(x - x_1) + 2a(y - y_1) = 0$

This is the equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1).

Example: Find the equation of tangent and normal to $x^2 = 8y$ at $(4, 2)$.

Solution: We have

$$x^2 = 8y$$

Differentiating w.r.t x

$$2x = 8\frac{dy}{dx}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{x}{4}}$$

Now slope of tangent at $(4, 2) = \left(\frac{dy}{dx}\right)_{(4,2)}$

i.e., $\boxed{m = 1}$

By point slope form the equation of tangent will be

$y - y_1 = m(x - x_1)$

$y - 2 = 1(x - 4) \qquad [\because (x_1, y_1) = (4, 2)]$

$\Rightarrow \quad x - y - 2 = 0$

$\because$ Normal is perpendicular to tangent

$\therefore$ Slope of normal $= m' = -1$

By point-slope form, equation of normal will be

$$y - 4 = -1(x - 2)$$

$$\Rightarrow \quad x + y - 6 = 0$$

9.5 Application of Parabola

Parabolas have important applications in suspension bridges, design of telescopes, radar antennas and lighting systems.

This is because of the important geometrical property of parabola which is stated as under:

Theorem: The tangent at a point P of parabola makes equal angles with the line through P parallel to the axis of parabola and the line through P and the focus.

Proof: Let line l be a tangent to parabola $y^2 = 4ax$ at point $P(x_1, y_1)$ as shown in Figure.

Let α be the angle between tangent and the line PQ parallel to the axis of the parabola and β be the angle between the tangent and the line through P and focus $F(a, 0)$.

$\because$ $P(x_1, y_1)$ lies on the parabola

$\therefore$ $y_1^2 = 4ax_1$

$\Rightarrow$ $a = \dfrac{y_1^2}{4x_1}$

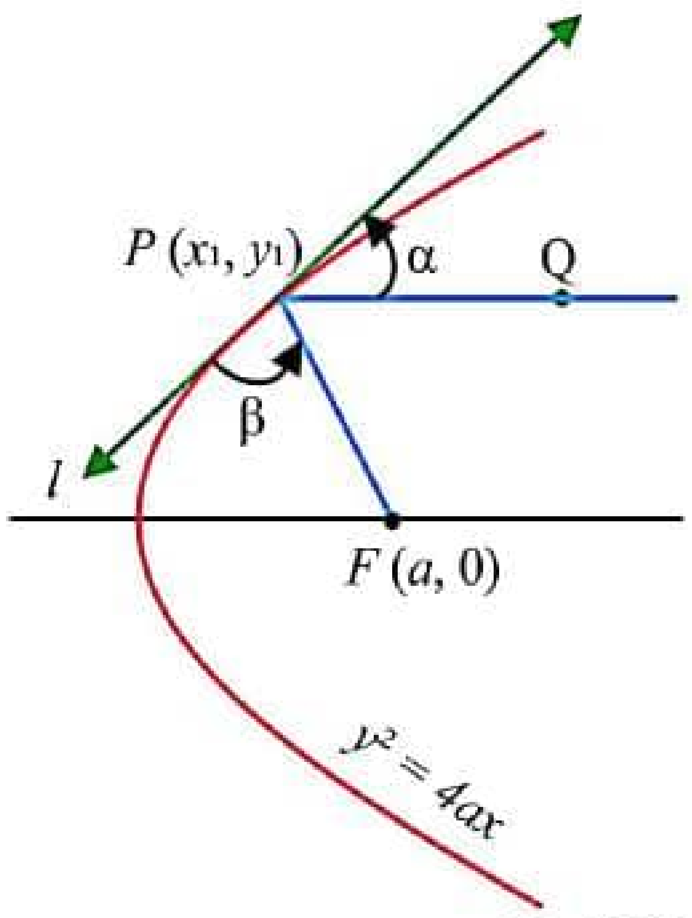


Fig 9.20

We have $y^2 = 4ax$

differentiating w.r.t x

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

Now, the slope of tangent to the parabola at P will be

$$m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{2a}{y_1}$$

$$= \frac{2}{y_1}\left(\frac{y_1^2}{4x_1}\right)$$

$$= \frac{y_1}{2x_1}$$

Slope of $\overleftrightarrow{PQ} = m_2 = 0$ and slope of $\overline{PF} = m_3 = \dfrac{y_1}{x_1 - a}$

$$= \frac{y_1}{x_1 - \dfrac{y_1^2}{4x_1}}$$

$$= \frac{4x_1y_1}{4x_1^2 - y_1^2}$$

Angle from $\overleftrightarrow{PQ}$ to the tangent

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\frac{y_1}{2x_1} - 0}{1 + 0}$$

$$\tan\alpha = \frac{y_1}{2x_1} \quad \ldots \text{(i)}$$

Angle from tangent to $\overline{PF}$

$$\tan\beta = \frac{m_3 - m_1}{1 + m_1 m_3}$$

$$= \frac{\frac{4x_1y_1}{4x_1^2 - y_1^2} - \frac{y_1}{2x_1}}{1 + \frac{4x_1y_1}{4x_1^2 - y_1^2} \times \frac{y_1}{2x_1}}$$

$$= \frac{8x_1^2y_1 - 4x_1^2y_1 + y_1^3}{8x_1^3 - 2x_1y_1^2 + 4x_1y_1^2}$$

$$= \frac{4x_1^2y_1 + y_1^3}{8x_1^3 + 2x_1y_1^2}$$

$$= \frac{y_1(4x_1^2 + y_1^2)}{2x_1(4x_1^2 + y_1^2)}$$

$$\tan\beta = \frac{y_1}{2x_1} \quad \ldots \text{(ii)}$$

$\because$ α and β are acute angles and $\tan\alpha = \tan\beta$ (from (i) and (ii))

$\therefore$ $\alpha = \beta$ Hence proved.

9.5.1 Solve suspension and reflection problems related to parabola

In physics, according to the law of reflection of light, the angle of incidence is equal to the angle of reflection at point P of the surface as shown in the Fig. 9.21.

i.e., $\theta_1 = \theta_2$

So, $\alpha = \beta$ (complements of congruent angles)

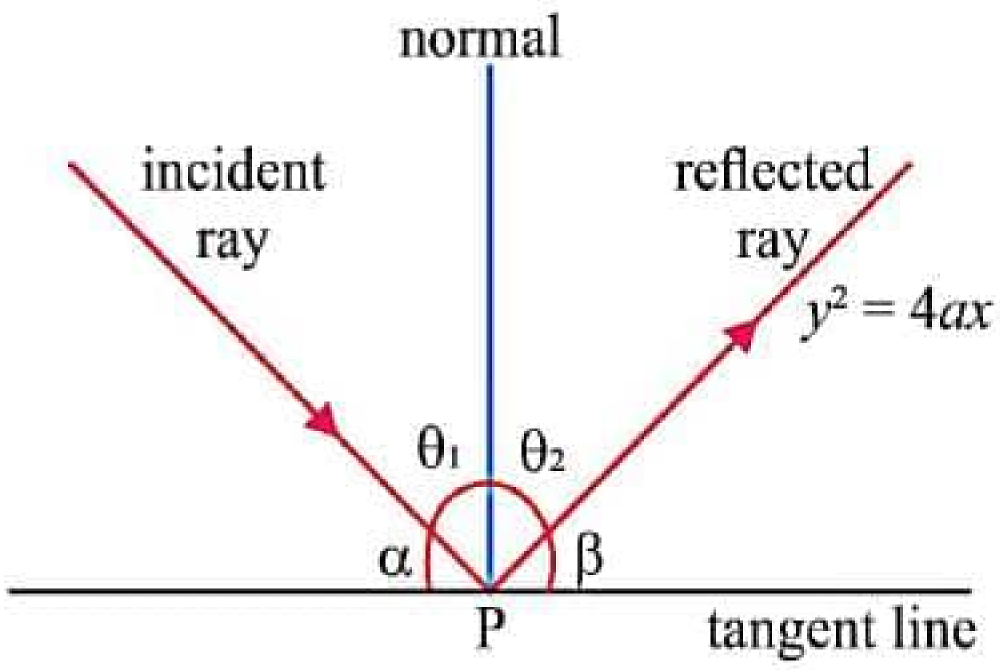


Fig 9.21

It means the angle between incident ray and the tangent line at P is equal to the angle between the reflected ray and the tangent line at P. Therefore, if the reflecting surface has parabolic cross sections with a common focus, then all light rays entering parallel to the axis of parabola will be reflected through the focus as shown in the figure 9.22.

In reflecting telescopes, this rule is used to reflect the parallel rays of light from the stars or planets off a parabolic mirror to an eye piece at the focus of the parabola.

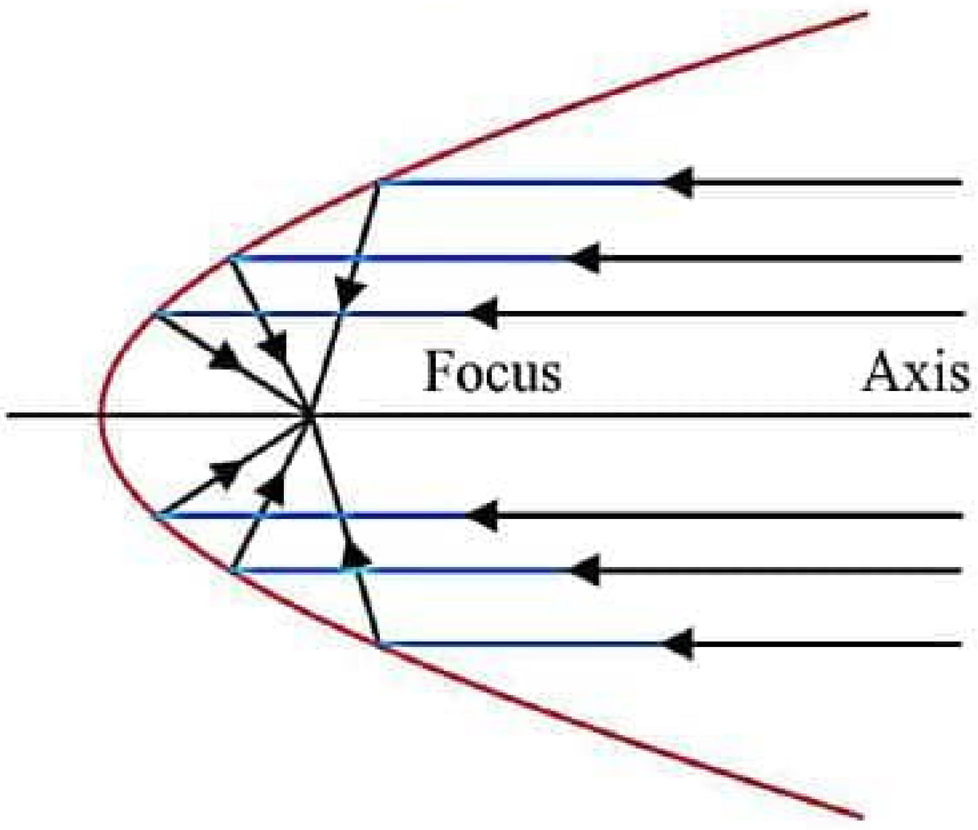


Fig. 9.22

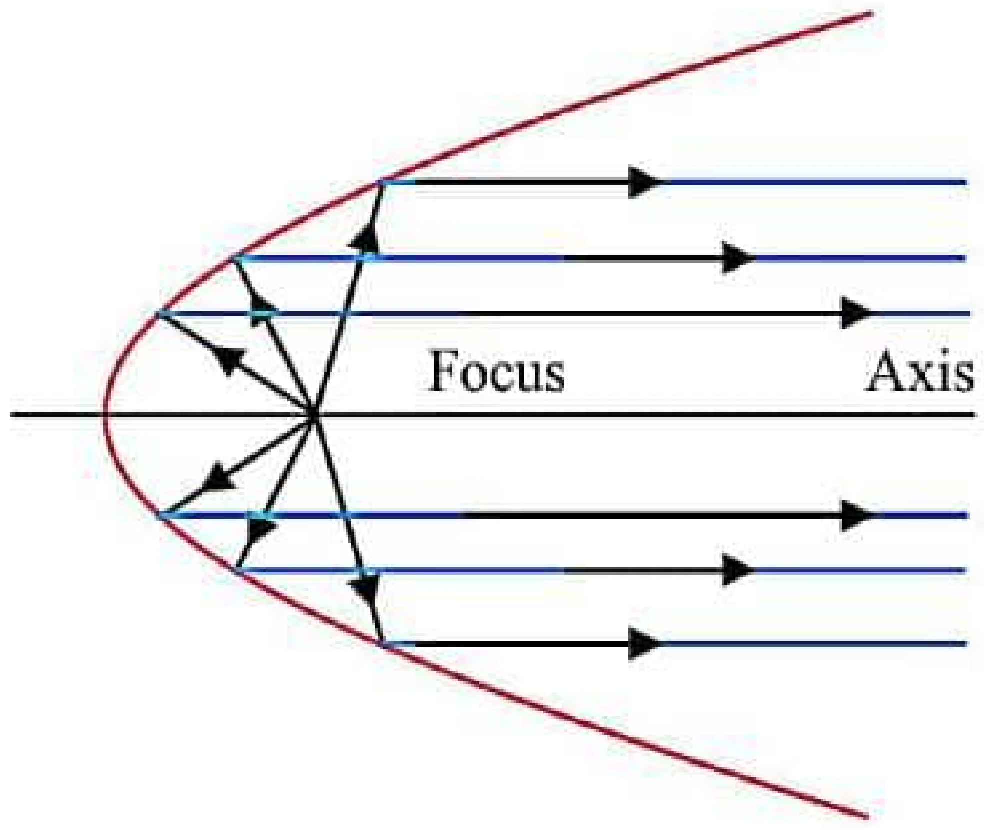


Fig. 9.23

Conversely, if a light source is located at the focus of a parabolic reflector, then the reflected rays will form a beam of parallel rays parallel to the axis of parabola as shown in the figure 9.23. The parabolic reflectors in automobile headlights and flash light use this rule. The optical principles which have been discussed above are also valid for radar signals, sound waves, radio waves etc.

Parabolas are also used in suspension problems related to suspension bridges and structures.

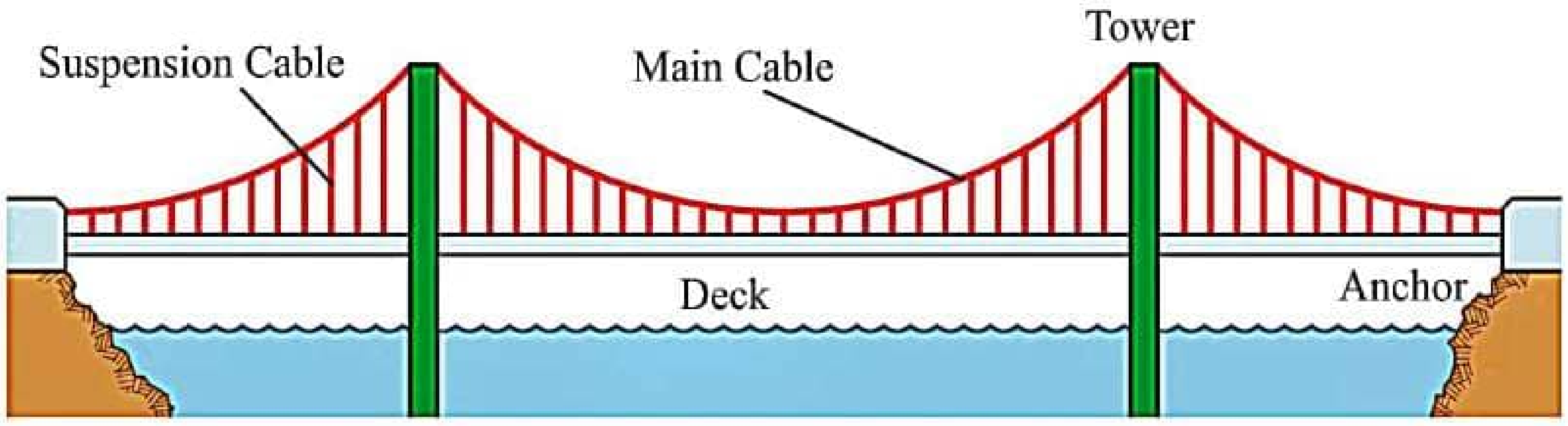


Fig. 9.24

We know that the cables of suspension bridges are mostly parabolic in shape. This shape provides the stability of bridges. The weight of the bridge and other physical forces (tensions, compressions) acting on the cable are transferred by the parabolic cables to the towers to which the cables are attached. This transfer of physical forces helps the bridges to remain operational for a long period of time.

Let us solve few examples related to suspension and reflection.

Example 1. How far from vertex should a light source be placed on the axis of parabolic reflector so that it produces a beam of parallel rays, whereas the depth and length of chord perpendicular on axis of parabolic reflector are 10 cm and 12 cm respectively and the parabola is cup-right.

Solution: Let the vertex of parabolic reflector is at origin as shown in the figure 9.25.

According to the condition $|\overline{AB}| = 12\ cm$ and $|\overline{OC}| = 10\ cm$.

$\because$ point C is the mid-point of $\overline{AB}$

$\therefore$ $|\overline{AC}| = |\overline{BC}| = 6\ cm$

Hence coordinates of A and B are $(10, 6)$ and $(10, -6)$ respectively.

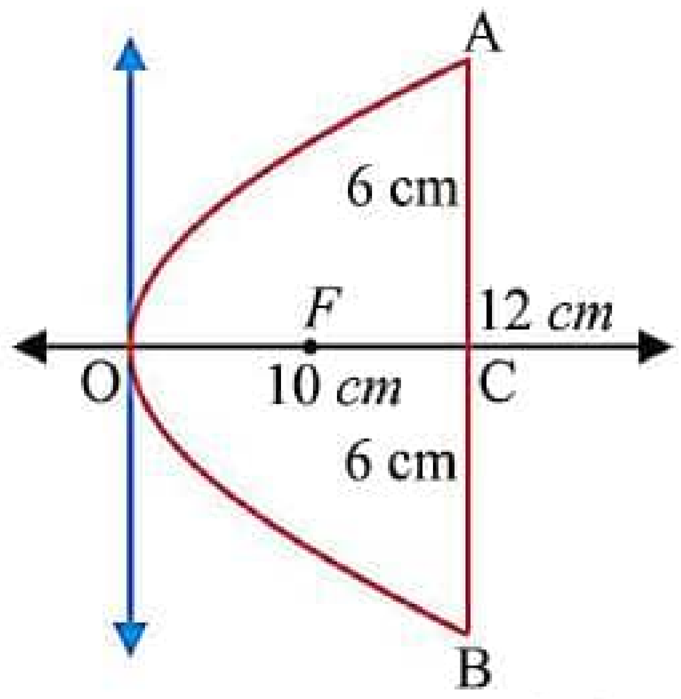


Fig 9.25

Le the equation of parabola be

$y^2 = 4ax$ ...(i)

$\because$ $A(10, 6)$ lies on the parabola

$\therefore$ we have $36 = 40a$

$\Rightarrow$ $a = \dfrac{9}{10}$

Now, focus $= (a, 0) = \left(\frac{9}{10}, 0\right)$

So, the light source should be placed at $F\left(\left(\frac{9}{10}, 0\right)\right)$ which is at a distance of $\frac{9}{10}$ cm from vertex.

Example 2. A main cable of a suspension bridge is suspended in the shape of parabola between two towers that are 600 ft apart and 90 ft above the roadway. If cable is at the height of 10 ft from the roadway at the centre of bridge then find:

(i) equation of parabola

(ii) height of suspender cable which is 150 ft away from the centre of bridge.

Solution: Let $\overline{AC}$ and $\overline{BD}$ represent towers of suspension bridge as shown in the Fig. 9.26.

According to the condition $|\overline{QE}| = 10, |\overline{QP}| = 80, |\overline{BD}| = 90$ and $|\overline{DE}| = 300$.

Let vertex Q is on y-axis then equation of parabola will be

$x^2 = 4a\,(y - 10)$ ...(i)

According to the condition, point $B(300, 90)$ lies on parabola.

So from equation (i), we get

$90000 = 4a(80)$

$\Rightarrow$ $a = \dfrac{9000}{32}$

$\Rightarrow$ $a = \dfrac{1125}{4}$

So, equation (i) becomes

$x^2 = 1125\,(y - 10)$ ...(ii)

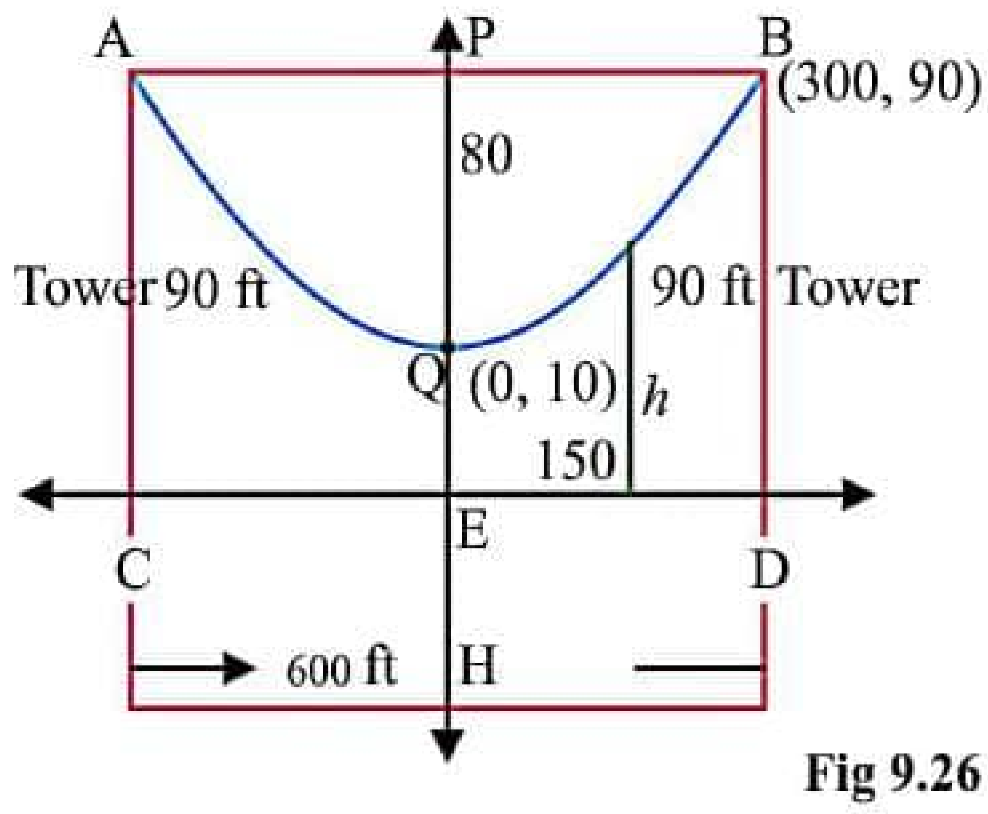


Fig 9.26

Let h be the height of suspender cable at 150 ft away from the centre of bridge.

$\because$ $T(150, h)$ lies on the parabola

$\therefore$ equation (ii) becomes

$(150)^2 = 1125\ (h - 10)$

$\frac{22500}{1125} = h - 10$

$\Rightarrow$ $h - 10 = 20$

$\Rightarrow$ $h = 30$

So, the required height is 30 ft.

Exercise 9.2

1. Find the condition when the line $y = mx + c$ is tangent to the parabola $x^2 = 4ay$. Also find point of contact and the equation of tangent.
2. Find condition of tangency and the point of tangency for the following lines and parabolas. Also find equation of tangent in each case:
   (i) $2x + y = c$; $y^2 = 10x$
   (ii) $3x + 4y = p$; $x^2 = 12y$
   (iii) $y = cx$; $y^2 = 8(x - 1)$
3. Find the equation of tangent and normal to the following parabolas at the given points:
   (i) $y^2 = 8x$; $(2, 4)$
   (ii) $x^2 = 4y$; $(-6, 9)$
   (iii) $(y - 1)^2 = 9(x - 2)$; $(3, 4)$
4. Find the equation of tangent and normal at $P(x_1, y_1)$ to the parabola $x^2 = 4ay$.
5. A light house uses a parabolic reflector that is 1 m in diameter. How deep should the reflector be if light source is placed halfway between the vertex and the plane of rim to produce parallel beam of light to the axis of parabola.
6. There is a parabolic reflector of 12 cm in diameter is used in a vehicle where should the light source be placed to produce parallel beam of light whereas the reflector is 8 cm deep.
7. The main cable of suspension bridge is suspended in the shape of parabola between two towers that are 100 m apart and 30 m high from the roadway. If the cable is at the height of 5 m from the roadway at the centre of the bridge then find the equation of parabola and the distance of 10 m high suspended cable from the centre of the bridge.
8. The main cable of a suspension bridge is in the shape of a parabola. The towers are 600 feet apart and 60 feet high from the roadway. If the cable touches at the roadway at the midway between the towers. What is height of the suspender cable 150 feet from the centre of the bridge.

9.6 Ellipse

We have already studied about ellipse that ellipse is an special type of conic. Here we will discuss its definition and elements in detail.

9.6.1 Define ellipse and its elements (i.e., centre, foci, vertices, covertices, directrices, major and minor axes, eccentricity, focal chord and latera recta)

An ellipse is defined on the basis of two geometrical properties, one is called focus-directrix property and the other one is related to the distances of a point of ellipse to two fixed points.

Definition 1: An ellipse is a set of all the points in plane whose distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is called focus, the fixed line is called directrix and the constant ratio is called eccentricity. We denote eccentricity by e whereas $0 < e < 1$. By symmetry ellipse has two foci and two directrices as shown in the figure 9.27. In the figure F_1 and F_2 are two foci whereas l_1 and l_2 are two directrices. The mid-point of the foci is called centre of the ellipse. The chord through two foci and the centre is called major axis whereas a chord through centre and perpendicular to the major axis is called minor axis. In the figure point C, $\overline{A_1A_2}$ and $\overline{B_1B_2}$ are the centre, major axis and minor axis of the ellipse respectively. The end points of major axis and minor axis are called vertices and covertices respectively.

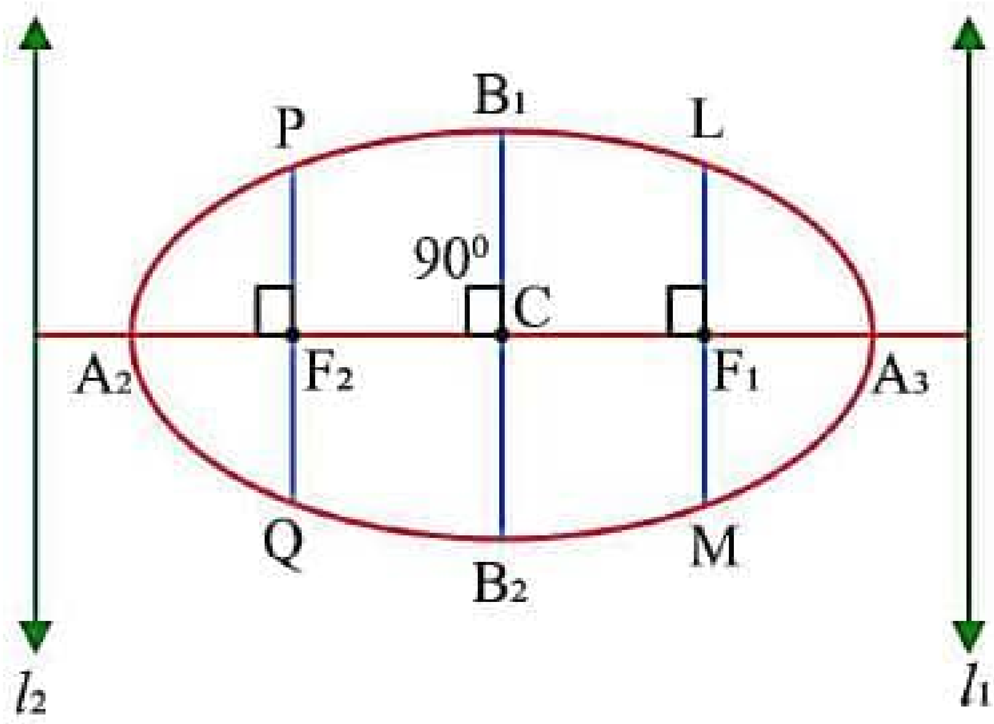


Fig 9.27

In the figure, A_1 and A_2 are vertices whereas B_1 and B_2 are covertices. Any chord through a focus is called focal chord of the ellipse whereas the focal chord which is perpendicular to the major axis is called latus rectum of the ellipse. In the figure $\overline{LM}$ and $\overline{PQ}$ are latera recta (plural of latus rectum) of the ellipse.

> **Note:** The major and minor axes together are called principal axes and their halves are called semi-axes.

Definition 2: An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points is a positive constant that is greater than the distance between the fixed points and equal to the length of major axis. The fixed points are called foci as shown in the figure 9.28.

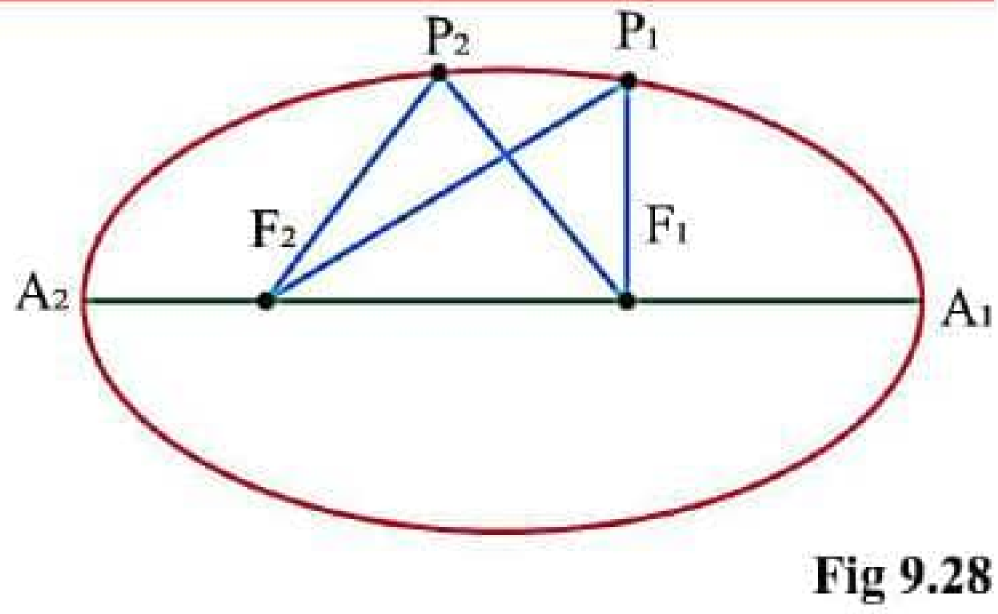


Fig 9.28

Let P_1 and P_2 be any two points of ellipse whereas F_1 and F_2 are foci as shown in the figure then by definition.

$$|\overline{P_1F_1}| + |\overline{P_1F_2}| = |\overline{P_2F_1}| + |\overline{P_2F_2}| = k$$

Where $k > 0$, $k > |\overline{F_1F_2}|$ and $k = |\overline{A_1A_2}|$

Basic relation of distances of focus, vertex and covertex from the centre of ellipse.

Let a, b, c are respectively the distance of vertex, covertex and focus from centre of the ellipse as shown in the figure 9.29. The basic relation of a, b, c is $a^2 = b^2 + c^2$. Let us prove it. First of all, we take two points P and Q such that P is at vertex and Q is at covertex.

According to the definition 2 of ellipse

$$|\overline{PF_1}| + |\overline{PF_2}| = |\overline{QF_1}| + |\overline{QF_2}|$$

i.e., $(a - c) + (a + c) = \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2}$

$\Rightarrow \quad 2a = 2\sqrt{b^2 + c^2}$

Squaring both sides

we get $a^2 = b^2 + c^2$.

Hence proved.

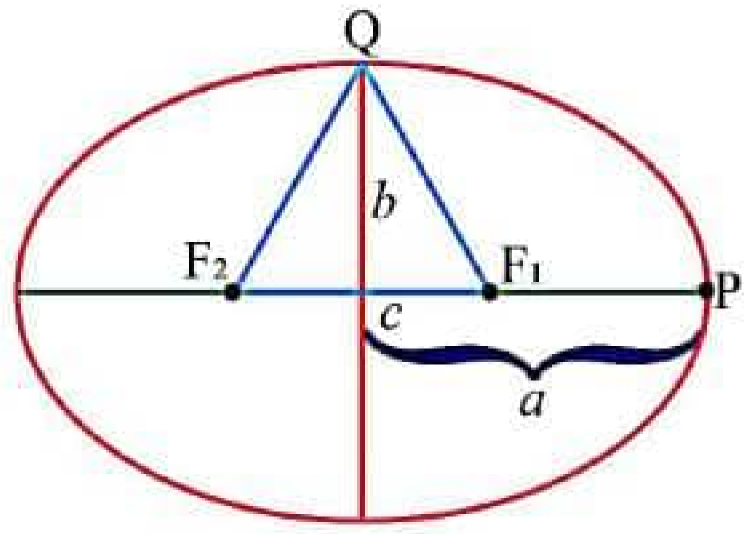


Fig 9.29

Relation of a, c and e where e is eccentricity of the ellipse.

Consider an ellipse whose centre is at C and directrices l_1 and l_2 as shown in the figure 9.30. A_1 and A_2 are two vertices and F_1 and F_2 are the foci. By definition of eccentricity

$$e = \frac{m\overline{A_1F_1}}{m\overline{A_1D_1}}$$

$\Rightarrow \quad m\overline{A_1F_1} = e(m\overline{A_1D_1}) \quad \ldots$(i)

Similarly, $m\overline{A_2F_1} = e(m\overline{A_2D_1}) \quad \ldots$(ii)

From Fig. 9.30

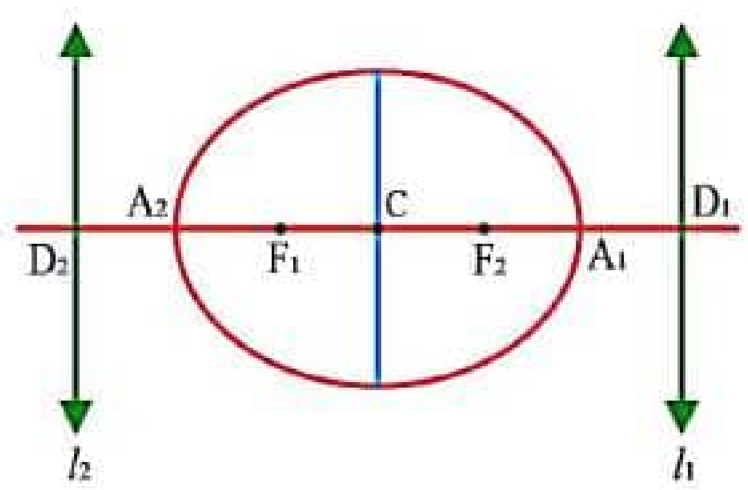


Fig 9.30

$m\overline{A_1A_2} = m\overline{A_2F_1} + m\overline{A_1F_1}$

or $\quad 2a = e(m\overline{A_2D_1} + m\overline{A_1D_1})$

$\Rightarrow \quad 2a = e(m\overline{A_2C} + m\overline{CD_1} + m\overline{CD_1} - m\overline{A_2C})$

$\Rightarrow \quad 2a = e(2m\overline{CD_1})$

$\Rightarrow \quad m\overline{CD_1} = \frac{a}{e} \quad \ldots$(iii)

Now, $\quad m\overline{CF_1} = m\overline{CA_1} - m\overline{A_1F_1}$

i.e., $\quad c = a - e\, m\overline{A_1D_1} \quad$ (using (i))

$c = a - e(m\overline{CD_1} - m\overline{CA_1})$

$\Rightarrow \quad c = a - e\left(\frac{a}{e} - a\right) \quad$ (using (iii))

$\Rightarrow \quad c = a - e\left(\frac{a - ae}{e}\right)$

$\Rightarrow \quad c = ae$

or $\quad \boxed{e = \frac{c}{a}}$

i.e., eccentricity of ellipse is also the ratio of distances of focus and vertex from centre.

9.6.2 Explain that circle is a special case of an ellipse

We know that in ellipse, the eccentricity "*e*" is given by

$$e = \frac{c}{a} \quad \text{where } 0 \leq e < 1$$

and it is the measure of the flatness of ellipse.

If we keep major axis constant then the closer the eccentricity is to 1, the flatter will be the ellipse. Conversely if *e* gets closer to zero the ellipse will become circle, as shown in the figure 9.31.

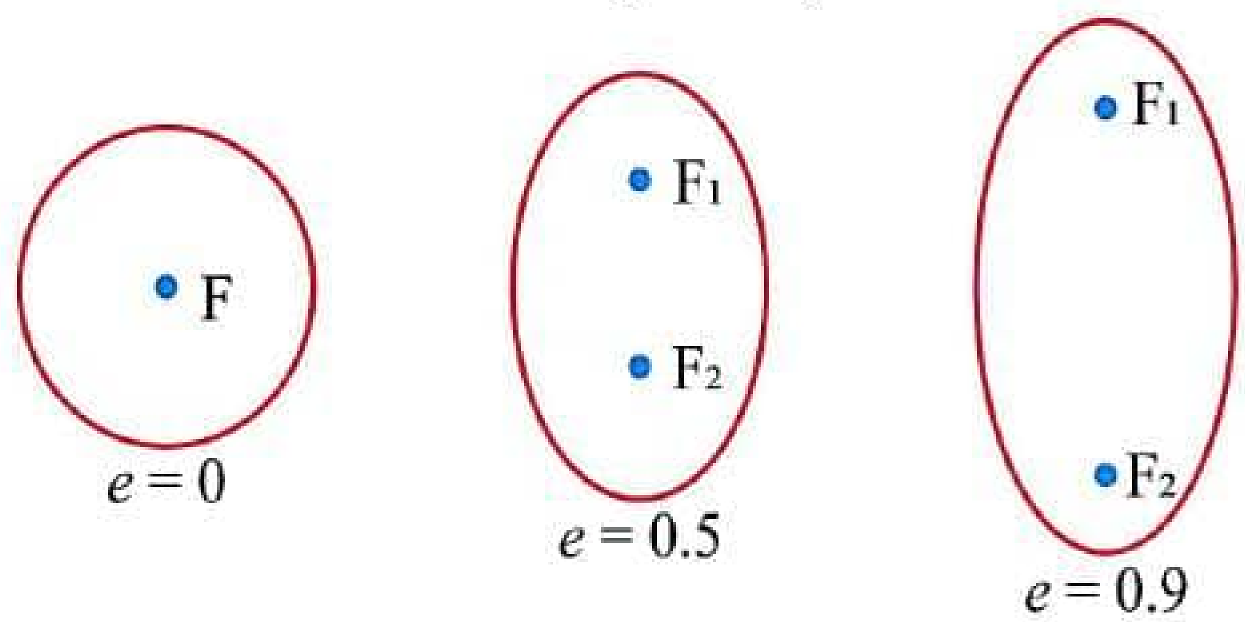


Fig 9.31

We have

$$e = \frac{c}{a}$$

If *c* approaches to zero then eccentricity will be zero and two foci will coincide and the resulting ellipse will be a circle.

We also have

$$a^2 = b^2 + c^2$$

If $\quad c = 0$

then $\quad a^2 = b^2$

i.e., $\quad a = b$

Hence circle is an special case of circle when eccentricity is zero, foci coincide and $a = b$.

9.7 Standard Form of Equation of an Ellipse

The simplest equations of ellipse are obtained when coordinate axes are positioned in such a way that the centre of ellipse is at the origin and the foci are on either *x*-axis or *y*-axis. The two possible such orientations are shown in the figure 9.32 and 9.33. These are called the standard positions of ellipse and their equations are called standard equations of ellipse.

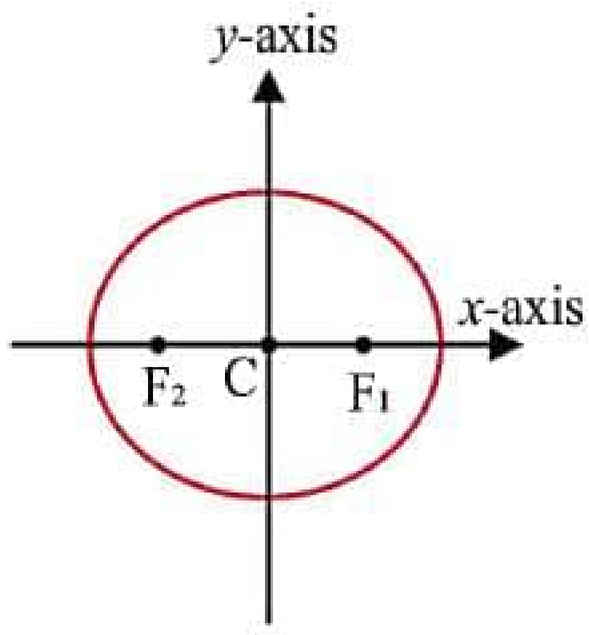


Fig 9.32

9.7.1 Derive the standard form of equation of an ellipse and identify its elements

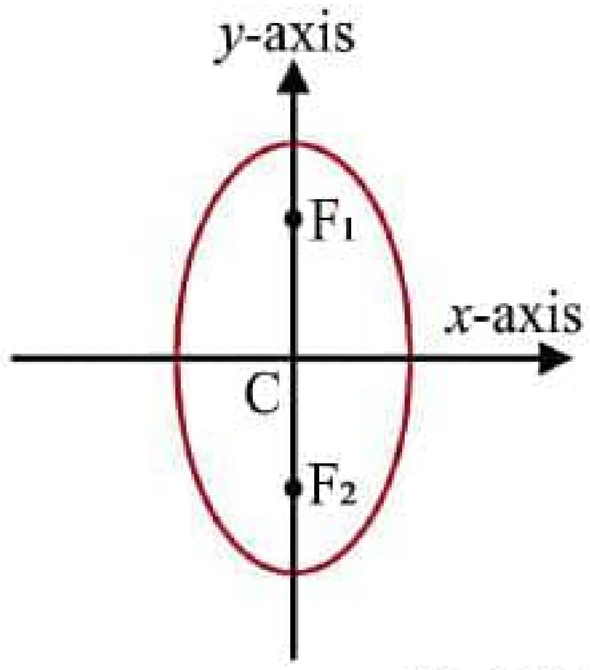


Fig 9.33

There are two standard forms of equation of ellipse: one when major axis is along x-axis and the other when major axis is along y-axis. We derive both standard forms.

(a) Standard form of equation of ellipse when major axis is along x-axis

Let $P(x,y)$ be any point of ellipse with major axis along x-axis and centre at origin as shown in the figure 9.34. Let a,b,c be the distances of vertex, covertex and focus from the centre respectively.

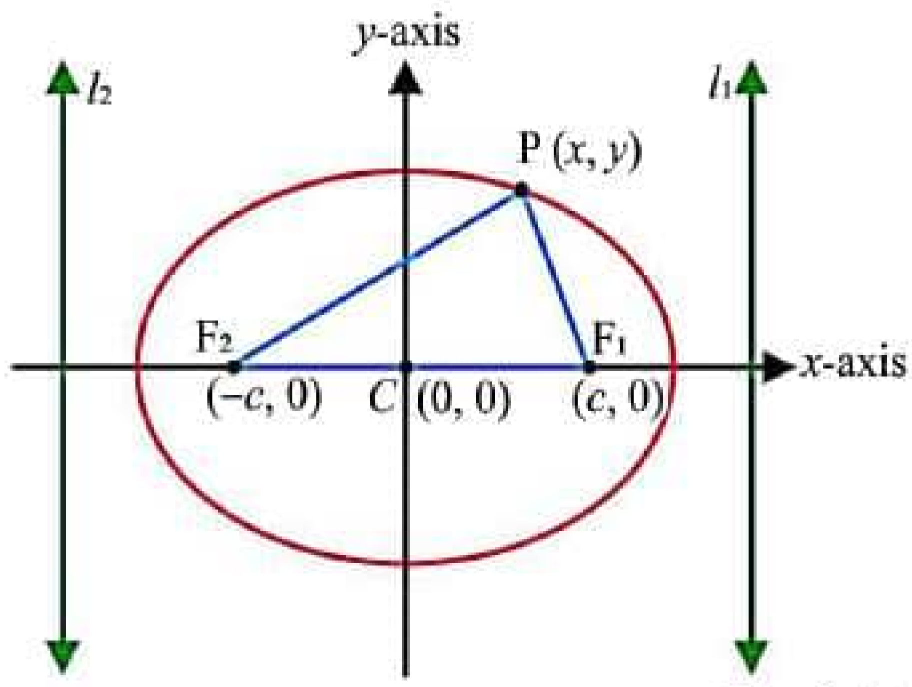


Fig 9.34

$\because$ foci are on x-axis

$\therefore$ foci are $F_1(c,0)$ and $F_2(-c,0)$

By the definition 2 of ellipse

$|\overline{PF_1}| + |\overline{PF_2}| = 2a$ where $a > b$

$\Rightarrow \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$

or $\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

Squaring both sides

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

or $a\sqrt{(x-c)^2+y^2} = a^2 - cx$

Again, squaring both sides

$$a^2\{(x-c)^2+y^2\} = a^4 - 2a^2cx + c^2x^2$$

$\Rightarrow (a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$

Dividing both sides by $a^2(a^2-c^2)$

we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2-c^2} = 1$

i.e., $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\qquad (\because\ a^2 = b^2 + c^2$ and $a > b)$

This is the standard equation of ellipse when major axis is along x-axis where coordinates of vertices and covertices are $(\pm a, 0)$ and $(0, \pm b)$ respectively.

We have already proved in section 9.6.1 that directrix is at distance of $\dfrac{a}{e}$ from the centre. So, equations of directrices are: $x = \pm\dfrac{a}{e}$.

(b) Standard form of equation of ellipse when major axis is along y-axis

Let $P(x, y)$ be any point of ellipse with major axis along y-axis and centre at origin as shown in the figure 9.35. Let a, b, c be the distances of vertex, covertex and focus from the centre respectively.

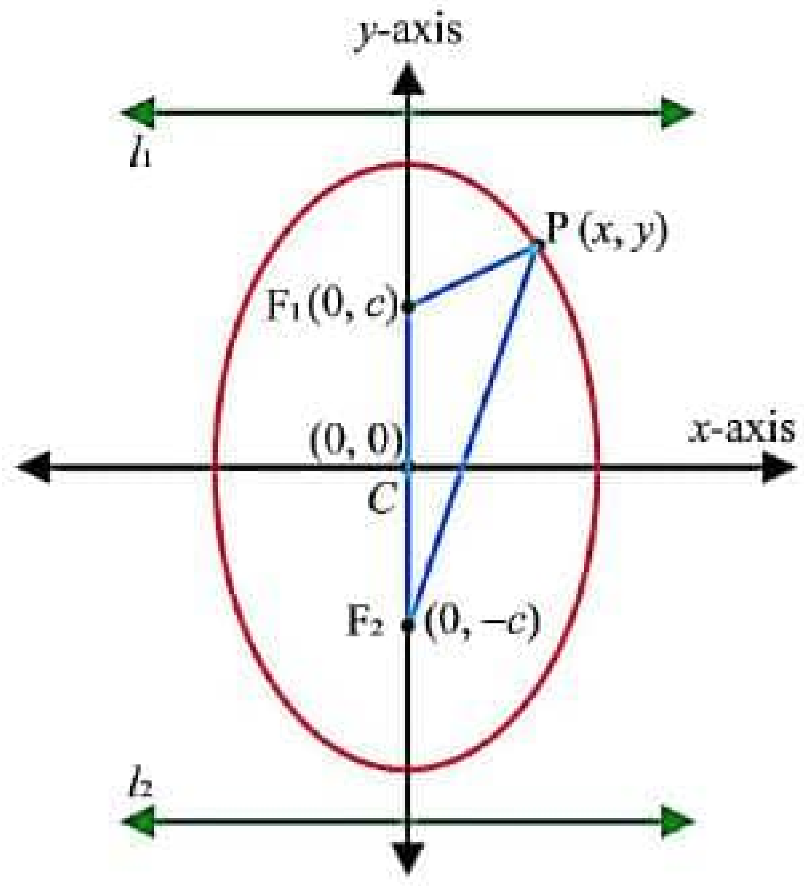


Fig 9.35

$\because$ foci are on y-axis

$\therefore$ foci are $F_1(0, c)$ and $F_2(0, -c)$

By the definition 2 of ellipse

$$|\overline{PF_1}| + |\overline{PF_2}| = 2a \quad \text{where } a > b$$

$$\Rightarrow \quad \sqrt{x^2 + (y-c)^2} + \sqrt{x^2 + (y+c)^2} = 2a$$

or $$\sqrt{x^2 + (y+c)} = 2a - \sqrt{x^2 + (y-c)^2}$$

Squaring both sides

$$(y+c)^2 + x^2 = 4a^2 - 4a\sqrt{x^2 + (y-c)^2} + x^2 + (y-c)^2$$

or $$a\sqrt{x^2 + (y-c)^2} = a^2 - cy$$

Again, squaring both sides

$$a^2\{x^2 + (y-c)^2\} = a^4 - 2a^2cy + c^2y^2$$

$$\Rightarrow \quad a^2x^2 + (a^2 - c^2)y^2 = a^2(a^2 - c^2)$$

Dividing both sides by $a^2(a^2 - c^2)$

We get $$\frac{x^2}{a^2 - c^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (\because \ a^2 = b^2 + c^2 \text{ and } a > b)$$

This is the standard equation of ellipse when major axis is along y-axis where coordinates of vertices and covertices are $(0, \pm a)$ and $(\pm b, 0)$ respectively.

We have already proved in section 9.6.1 that directrix is at distance of $\frac{a}{e}$ from the centre. So, equations of directrices will be: $y = \pm\frac{a}{e}$.

Length of latus rectum:

Let $\overline{AB}$ be the latus rectum of ellipse with major axis along x-axis having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (i)}$$

$\because$ Focus is on x-axis

$\therefore$ Coordinates of one focus are $(c, 0)$

Now equation of line containing latus rectum is

$x = c$ or $x = ae$ $(\because c = ae)$

By using $x = ae$ in equation (i)

We get, $\dfrac{a^2e^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\Rightarrow \quad y^2 = b^2(1 - e^2)$

$\Rightarrow \quad y = \pm b\sqrt{1 - e^2}$

So, the end points of latus rectum are

$A\left(c, b\sqrt{1-e^2}\right)$ and $B\left(c, -b\sqrt{1-e^2}\right)$

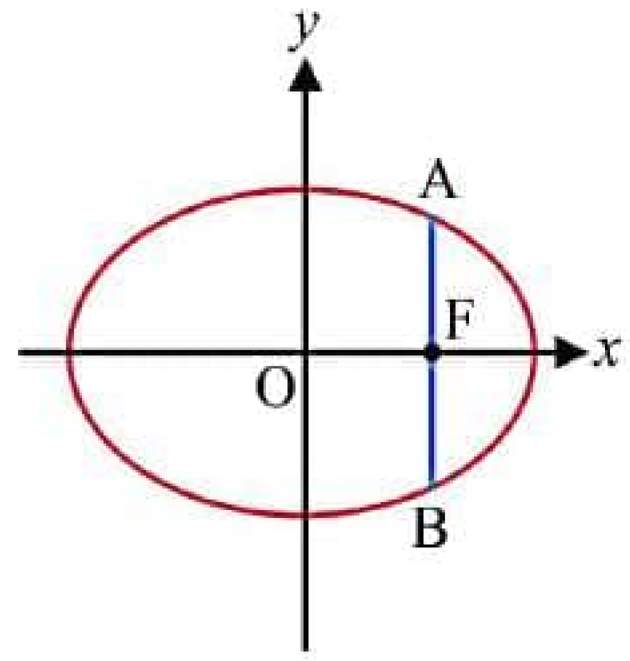


Fig 9.36

Now, $m\overline{AB} = 2b\sqrt{1 - e^2}$

$$= 2b\sqrt{\frac{a^2 - c^2}{a^2}} \qquad \left(\because\ e = \frac{c}{a}\right)$$

$$= 2b\left(\frac{b}{a}\right) \qquad (\because\ a^2 - c^2 = b^2)$$

$$= \frac{2b^2}{a}$$

So, the length of latus rectum is $\dfrac{2b^2}{a}$.

Standard equations of translated ellipses:

If the axes of an ellipse are parallel to the coordinate axes and centre is not at the origin then, by the translation the equations of ellipse may be determined which are as under:

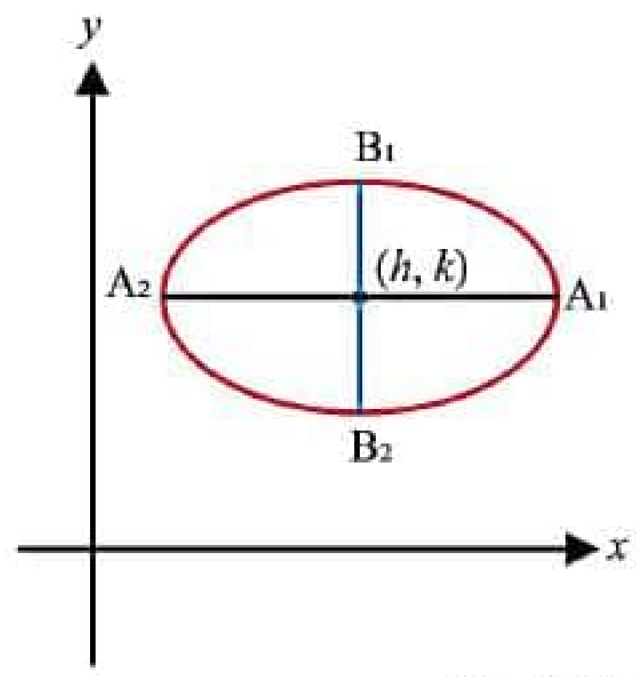


Fig 9.37

(i) Ellipse with centre (h, k) and major axis parallel to x-axis.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\ (a > b)$$

In this case foci are $(h \pm c, k)$, vertices $(h \pm a, k)$, covertices are $(h, k \pm b)$ and directrices are $x - h = \pm\frac{a}{e}$.

(ii) Ellipse with centre (h, k) and major axis parallel to y-axis.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1\ (a > b)$$

In this case foci are $(h, k \pm c)$, vertices are $(h, k \pm a)$ covertices are $(h \pm b, k)$ and directrices are $y - k = \pm\frac{a}{e}$.

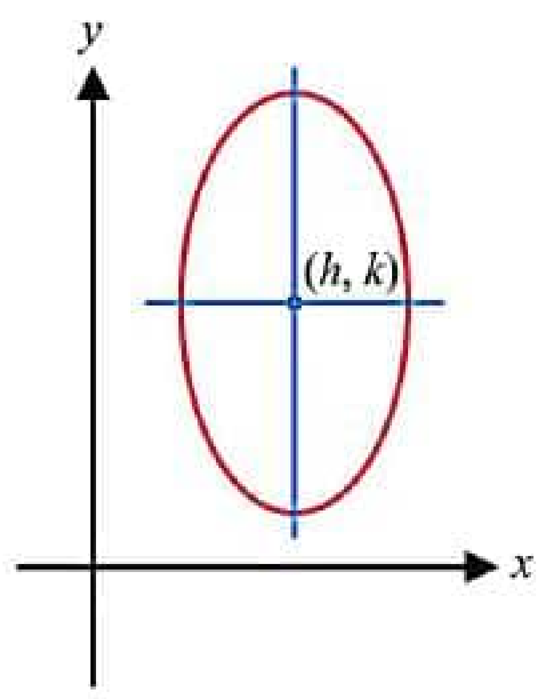


Fig 9.38

General equation of ellipse when axes of ellipse are parallel to coordinate axes

Consider a translated ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$\Rightarrow \quad b^2(x^2-2hx+h^2)+a^2(y^2-2ky+k^2)=a^2b^2$

$\Rightarrow \quad b^2x^2+a^2y^2-2hb^2x-2ka^2y+b^2h^2+a^2k^2-a^2b^2=0 \quad \ldots$(i)

Let $\quad A=b^2, B=a^2$

$G=-2hb^2, H=-2ka^2$ and $C=b^2h^2+a^2k^2-a^2b^2$

then equation (i) becomes

$Ax^2+By^2+Gx+Hy+C=0$, where A and B are non-zero with same sign.

This is general equation of ellipse when axes of ellipse are parallel to coordinate axes.

Summary of equations of ellipse

Equation	Related terms and conditions
(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$	Major axis is along x-axis and centre is origin Foci: $(\pm c, 0)$, Vertices: $(\pm a, 0)$ Covertices: $(0, \pm b)$ and Directrices: $x=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
(ii) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a>b$	Major axis is along y-axis and centre is at origin. Foci: $(0, \pm c)$, Vertices: $(0, \pm a)$ Covertices: $(\pm b, 0)$ and Directrices: $y=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
(iii) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1(a>b)$	Major axis is parallel to x-axis with centre at (h,k) Foci: $(h\pm c, k)$, Vertices: $(h\pm a, k)$ Covertices: $(h, k\pm b)$ and Directrices: $x-h=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$

Equation	Related terms and conditions
(iv) $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	Major axis is parallel to y-axis with centre at (h,k) Foci: $(h,k\pm c)$, Vertices: $(h,k\ \pm a)$ Covertices: $(h\pm b,k)$ and Directrices: $y-k=\pm\frac{a}{e}$ Length of latus rectum: $\frac{2b^2}{a}$
General equation $Ax^2+By^2+Gx+Hy+C=0$	A and B are non-zero with same signs. All related elements can be found by converting into a standard form.

Note: For all types of ellipses mentioned above, we have

$$a^2=b^2+c^2,\ \text{latus rectum}=\frac{2b^2}{a}\ \text{and}\ c=ae.$$

Auxiliary Circle: Auxiliary circle of ellipse is the circle whose diameter is the major axis of ellipse. For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, auxiliary circle is: $x^2+y^2=a^2$.

Example 1. Find foci, vertices, covertices, latus rectum and equations of directrices, of the ellipse: $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution: This is the ellipse with centre at origin and major axis along x-axis.

Here, $a^2=25$ and $b^2=9$

So, $a=5$ and $b=3$ are the semi-axes of the ellipse,

we know that

$$a^2=b^2=c^2$$

$\Rightarrow\quad 25=9+c^2$

$\Rightarrow\quad c^2=16$

$\Rightarrow\quad c=4$

We know that $c=ae$

$\Rightarrow\quad 4=5e \qquad \Rightarrow \qquad e=\frac{4}{5}$

So, the eccentricity is $\frac{4}{5}$.

Now,

Foci $=(\pm c,0)=(\pm 4,0)$,

Vertices $= (\pm a, 0) = (\pm 5, 0)$,

Covertices $= (0, \pm b) = (0, \pm 3)$,

Latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{5} = \frac{18}{5}$

and equation of directrices are

$$x = \pm\frac{a}{e}$$

i.e., $x = \pm\left(\frac{5}{\frac{4}{5}}\right)$

or $x = \pm\frac{25}{4}$.

Example 2. Find semi-axes, centre, foci, vertices, covertices, latus rectum and equations of directrices of the ellipse $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1$.

Solution: Comparing given ellipse with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

we get $a^2 = 25, b^2 = 16, h = 2, k = -3$ and major axis is parallel to y-axis.

We know that

$a^2 = b^2 + c^2$

$\Rightarrow \quad 25 = 16 + c^2$

$\Rightarrow \quad c^2 = 9$

$\Rightarrow \quad c = 3$

We know that $c = ae$

$\Rightarrow \quad 3 = 5e \quad \Rightarrow \quad e = \frac{3}{5}$

So, the eccentricity is $\frac{3}{5}$ and $a = 5, b = 4$ are the semi-axes of the ellipse.

Now,

Centre $= (h, k) = (2, -3)$

Foci $= (h, k \pm c) = (2, -3 \pm 3)$

So, Foci are $(2, 0)$ and $(2, -6)$

Vertices $= (h, k \pm a) = (2, -3 \pm 5)$

So, Vertices are $(2, 2)$ and $(2, -8)$

Covertices are $(h \pm b, k) = (2 \pm 4, -3)$,

So, Covertices are $(2, 1)$ and $(2, -7)$

Latus rectum $= \frac{2b^2}{a} = \frac{2\times 16}{5} = \frac{32}{5}$

and equation of directrices are

$$y - k = \pm\frac{a}{e}$$

$$y + 3 = \pm\left(\frac{5}{\frac{3}{5}}\right)$$

or $$y + 3 = \pm\frac{25}{3}.$$

9.7.2 Find the equation of an ellipse with the following given elements:

- **major and minor axes,**
- **two points,**
- **foci, vertices or lengths of a latera recta,**
- **foci, minor axes or length of a latus rectum.**

- **Equation of ellipse whose major and minor axes are given**

The method is explained with the help of the following example.

Example: Find the equation of ellipse with centre at origin where major and minor axes are 10 and 8 units respectively and major axis is along x-axis.

Solution: Here length of major axis = 10

i.e., $2a = 10$

$\Rightarrow$ $a = 5$

and length of minor axis = 8

i.e., $2b = 8$

$\Rightarrow$ $b = 4$

According to the condition, equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (i)} \quad (\because \text{ major axis is along } x\text{-axis})$$

By using values of a and b in equation (i)

we get,

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

This is the required equation of ellipse.

- **Equation of ellipse when two points are given**

The method is explained with the help of the following example.

Example: Find the equation of ellipse passing through $(1, \sqrt{2})$ and $\left(\frac{\sqrt{6}}{2}, 1\right)$ whereas the centre is at origin and major axis is along y-axis.

Solution:

$\because$ major axis is along y-axis and centre at origin

$\therefore$ equation of ellipse will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots(i)$$

$\because$ $(1, \sqrt{2})$ lies on the ellipse

$\therefore$ we have from equation (i)

$$\frac{1}{b^2} + \frac{2}{a^2} = 1 \qquad \ldots(ii)$$

$\because$ $\left(\frac{\sqrt{6}}{2}, 1\right)$ lies on the ellipse

$\therefore$ we have from equation (i)

$$\frac{3}{2b^2} + \frac{1}{a^2} = 1 \qquad \ldots(iii)$$

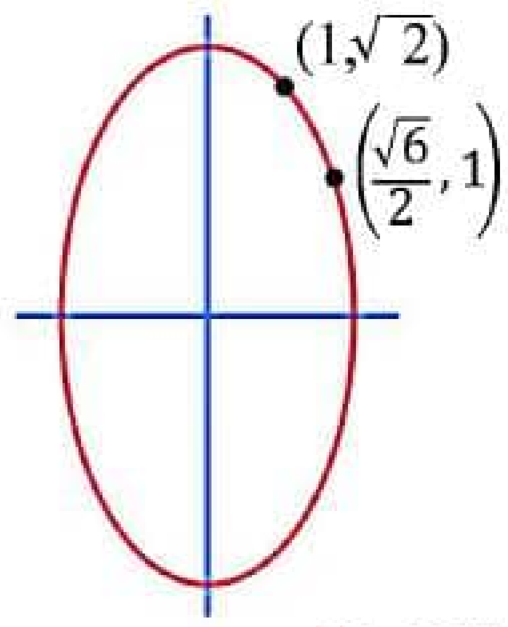


Fig 9.39

Multiplying equation (ii) by $\frac{3}{2}$ and subtracting equation (iii) from the resultant equation

$$\frac{3}{2b^2} + \frac{3}{a^2} = \frac{3}{2}$$

and $$\pm\frac{3}{2b^2} \pm \frac{1}{a^2} = -1$$

or $$\frac{2}{a^2} = \frac{1}{2}$$

$\Rightarrow$ $a^2 = 4$ $\Rightarrow$ $a = 2$

By using $a^2 = 4$ in (ii)

We get, $\frac{1}{b^2} + \frac{2}{4} = 1$

$\Rightarrow$ $\frac{1}{b^2} = \frac{1}{2}$ $\Rightarrow$ $b^2 = 2$

By using values of a and b in equation (i)

we get, $\frac{x^2}{2} + \frac{y^2}{4} = 1$

$\Rightarrow$ $4x^2 + 2y^2 = 8$

This is the required equation of ellipse.

- **Equation of ellipse when foci, vertices or length of latera recta are given**

  The method is explained with the help of the following examples.

Example 1. Find the equation of ellipse with centre at origin whose focus and vertex are $(8, 0)$ and $(10, 0)$ respectively.

Solution:

According to the condition, focus and vertex lie on x-axis.

So, major axis is along x-axis.

Here $c = 8$

$a = 10$

We know that

$$a^2 = b^2 + c^2$$

i.e., $100 = b^2 + 64$

$\Rightarrow$ $b^2 = 36$ $\Rightarrow$ $b = 6$

Equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e., $\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$

This is the required equation of ellipse.

Example 2. Find the equation of ellipse with centre at origin such that its focus is $(0, 3)$ and latus rectum is of $\frac{32}{5}$ units.

Solution:

According to the condition, major axis is along y-axis.

We have,

$c = 3$

and length of latus rectum $= \frac{32}{5}$

i.e., $\dfrac{2b^2}{a} = \dfrac{32}{5}$

$\Rightarrow$ $b^2 = \dfrac{16a}{5}$

We know that

$$a^2 = b^2 + c^2$$

i.e., $a^2 = \dfrac{16a}{5} + 9$

$\Rightarrow$ $5a^2 = 16a + 45$

$\Rightarrow$ $5a^2 - 16a - 45 = 0$

$\Rightarrow \quad 5a^2 - 25a + 9a - 45 = 0$

$\Rightarrow \quad 5a(a-5) + 9(a-5) = 0$

$\Rightarrow \quad (a-5)(5a+9) = 0$

$\Rightarrow \quad a = 5 \text{ or } a = -\frac{9}{5}$

$\because \quad a$ cannot be $-$ve

$\therefore \quad$ we neglect $a = -\frac{9}{5}$

Hence $a = 5$

We knot that

$$a^2 = b^2 + c^2$$

i.e., $\quad 25 = b^2 + 9$

$\Rightarrow \quad b^2 = 16 \quad \Rightarrow \quad b = 4$

Now equation of ellipse will be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

i.e., $$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

This is the required equation of ellipse.

- **Equation of ellipse when foci, minor axis or length of latus rectum are given**

The method is explained with the help of the following examples.

Example 1. Find the equation of an ellipse whose focus is $(5, 0)$ and minor axis is 12 units long and along y-axis where centre is at origin.

Solution:

According to the given conditions, equation of ellipse will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (i)}$$

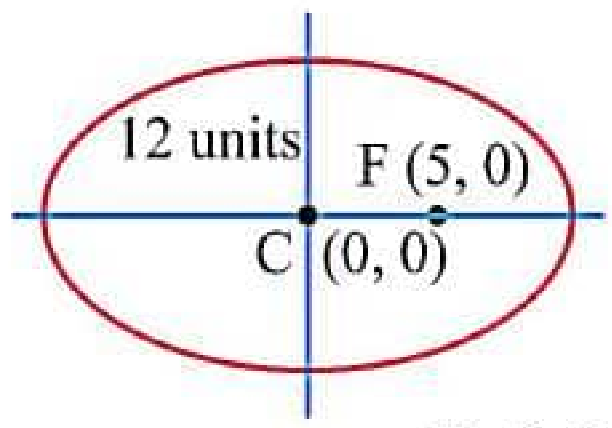


Fig 9.40

Here $\quad c = 5$

and length of minor axis = 10

i.e., $\quad 2b = 12$

$\Rightarrow \quad b = 6$

We know that

$$a^2 = b^2 + c^2$$

i.e., $\quad a^2 = 36 + 25$

$$a^2 = 61 \quad \Rightarrow \quad a = \sqrt{61}$$

By using values of a and b in equation (i)

we get

$$\frac{x^2}{61}+\frac{y^2}{36}=1$$

This is the required equation of ellipse.

Example 2. Find the equation of an ellipse whose minor axis is 10 units long and along x-axis whereas latus rectum is 8 units long and centre is at origin.

Solution:

According to the condition, the equation of ellipse will be

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Here length of minor axis $= 10$

i.e., $2b = 10 \quad \Rightarrow \quad b = 5$

and length of latus rectum $= 8$

i.e., $\frac{2b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

i.e., $a = \frac{25}{4}$

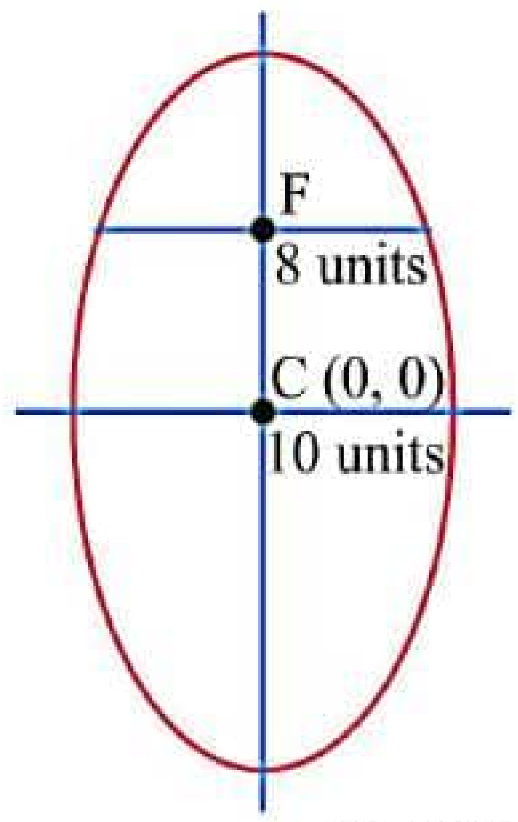


Fig 9.41

By using values of a and b in equation (i)

We get

$$\frac{x^2}{25}+\frac{y^2}{\frac{25}{4}}=1$$

$$\Rightarrow \quad \frac{x^2}{25}+\frac{4y^2}{25}=1$$

This is the required equation of ellipse.

9.7.3 Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph

In section 9.7.1, we have already studied the general equation of ellipse when major and minor axes of ellipses are parallel to the coordinate axes. The equation is as under:

$$Ax^2 + By^2 + Gx + Hy + C = 0$$

where A and B are non-zero and having same sign.

Equation (i) can be converted into standard form of equation of an ellipse. The method will be explained in the following examples.

Technique for drawing graph of an ellipse

Graph of an ellipse can be drawn from their standard equations using the following three steps:

1. Determine whether the major axis is along x-axis or y-axis or parallel to one of them. If denominator of x^2 or $(x-h)^2$ is larger then major axis is along x-axis or parallel to x-axis respectively. If denominator of y^2 or $(y-k)^2$ is larger then major axis is along y-axis or parallel to y-axis respectively. If both denominators are equal then it is a circle.
2. Determine the values of a and b and draw rectangle extending a units on each side of the centre along major axis and b units on each side of the centre along the minor axis.
3. Using the rectangle as guide, sketch the ellipse so that it touches the sides of the rectangle where the sides intersect the axes of the ellipse.

Example: Find foci, eccentricity, vertices, covertices, latus rectum and equations of directrices of ellipse $9x^2+16y^2-144=0$. Also draw its graph.

Solution: We have

$$9x^2+16y^2-144=0$$

or $\quad 9x^2+16y^2=144$

Dividing both sides by 144

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

Here $a^2=16$ and $b^2=9$, so $a=4$ and $b=3$ and major axis is along x-axis and centre at origin.

We know that

$$a^2=b^2+c^2$$

$$16=9+c^2 \quad \Rightarrow \quad c^2=7 \quad \text{or} \quad c=\sqrt{7}$$

Also, we know that

$$c=ae$$

So, $\quad \dfrac{\sqrt{7}}{4}=e$

Now,

$$\text{Foci}=(\pm c,0)=\left(\pm\sqrt{7},0\right)$$

So, $\quad$ Foci are $\left(\sqrt{7},0\right)$ and $\left(-\sqrt{7},0\right)$

Vertices $=(\pm a,0)$ and $(\pm 4,0)$

So, $\quad$ Vertices are $(4,0)$ and $(-4,0)$

Covertices $= (0, \pm b) = (0, \pm 3)$

So, Covertices are $(0, 3)$ and $(0, -3)$

Latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$

equations of directrices will be

$$x = \pm\frac{a}{e}$$

$$x = \pm\frac{4}{\frac{\sqrt{7}}{4}}$$

$$\Rightarrow \quad x = \pm\frac{16}{\sqrt{7}}.$$

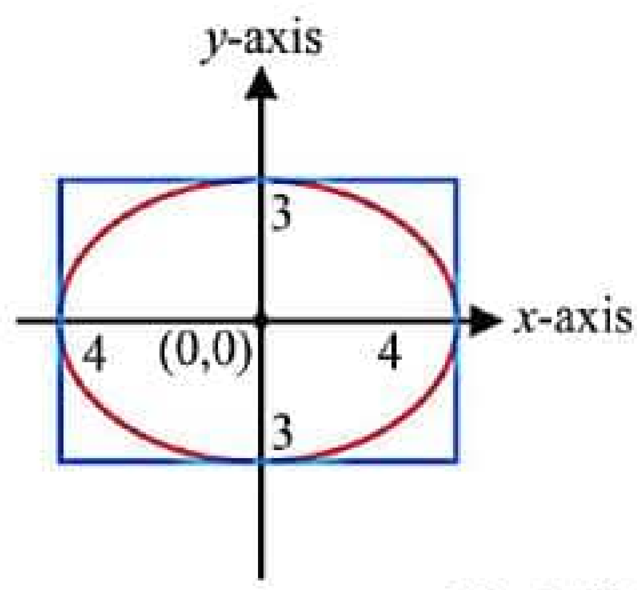


Fig 9.42

Graph of ellipse

Here centre $= (0, 0)$

Semi axes are $a = 4$ and $b = 3$

Major axis is along x-axis

Graph is shown in Fig. 9.42.

Example 2. Find centre, foci, vertices, and latus rectum of ellipse $9x^2 - 18x + 4y^2 + 16y - 11 = 0$. Also draw its graph.

Solution: We have

$$9x^2 - 18x + 4y^2 + 16y - 11 = 0$$

$$\Rightarrow \quad 9(x^2 - 2x) + 4(y^2 + 4y) = 11$$

$$\Rightarrow \quad 9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16$$

$$\Rightarrow \quad 9(x-1)^2 + 4(y+2)^2 = 36$$

Dividing both sides by 36

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

Comparing with $\frac{(x-h)^2}{b^2} + \frac{(y+k)^2}{a^2} = 1$

We get centre $= (h, k) = (1, -2)$

$$a^2 = 9 \text{ and } b^2 = 4$$

$$\Rightarrow \quad a = 3 \text{ and } b = 2$$

Also, major axis is parallel to y-axis.

We know that

$$a^2 = b^2 + c^2$$

i.e., $9 = 4 + c^2$

$\Rightarrow$ $c^2 = 5$ $\Rightarrow$ $c = \sqrt{5}$

Now, foci $= (h, k \pm c) = (1, -2 \pm \sqrt{5})$

So, foci are $(1, -2 + \sqrt{5})$ and $(1, -2, -\sqrt{5})$

Vertices $= (h, k \pm a) = (1, -2, \pm 3)$

So, Vertices are $(1, 1)$ and $(1, -5)$

Latus rectum $= \dfrac{2b^2}{a}$

$= \dfrac{2(4)}{9} = \dfrac{8}{9}$

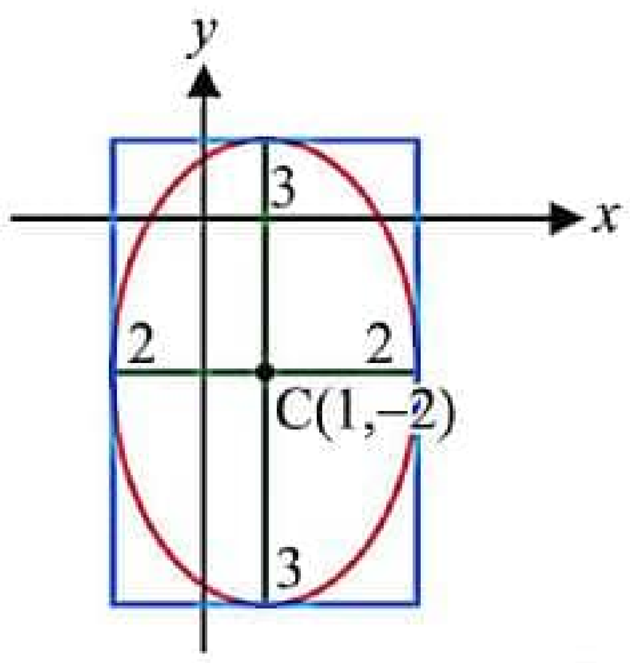


Fig 9.43

Graph of ellipse

Here centre $= (1, -2)$

Semi-axes are $a = 3$ and $b = 2$

Major axis is parallel to y-axis.

Graph of the ellipse is shown in the Fig. 9.43.

Exercise 9.3

1. Find semi-axes, eccentricity, foci, vertices, covertices, latus rectum and equations of directrices of the following ellipses. Also draw their graphs.

(i) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ (ii) $\dfrac{x^2}{16} + \dfrac{y^2}{10} = 1$

(iii) $\dfrac{(x-3)^2}{25} + \dfrac{(y+4)^2}{16} = 1$ (iv) $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{16} = 1$

(v) $9x^2 + 25y^2 = 225$ (vi) $4x^2 - 16x + 25y^2 + 200y - 316 = 0$

2. Find the equations of the following ellipse whose centres are at origin and their axes are along coordinate axes. Also satisfy the given conditions:

(i) Major and minor axes are 12 and 8 respectively with minor axis is along y-axis.

(ii) Ellipse passes through $\left(1, \sqrt{\dfrac{3}{2}}\right)$ and $\left(\dfrac{2}{\sqrt{3}}, 1\right)$ with major axis is along y-axis.

(iii) Foci at $(\pm 3, 0)$ and vertices at $(\pm 5, 0)$

(iv) Foci at $(0, \pm 4)$ and latus rectum $\dfrac{18}{5}$

(v) Foci at $(\pm 5, 0)$ and minor axis is 12 units long and along y-axis.

(vi) Minor axis along x-axis with length is 8 units and latus rectum are 6 units long and along y-axis.

(vii) Covertices at $(0, \pm 3)$ and distance between foci $= 10$ units.

(viii) Directrix $y = \frac{25}{3}$ and latus rectum $\frac{32}{5}$

3. Find equation of auxiliary circle to $5x^2 + 7y^2 = 11$.
4. Is the point $(4, 5)$ inside on or outside the ellipse $2x^2 + 3y^2 = 6$.
5. Find equation of ellipse with centre at $(5, -3)$, one vertex at $(10, -3)$ and one focus at $(9, -3)$.
6. If ellipse is $9x^2 + 13y^2 = 117$ then find:
   (i) distance between foci;
   (ii) distance between vertices;
   (iii) distance between covertices.
7. Find eccentricity of ellipse if:
   (i) axes are 32 and 24;
   (ii) latus rectum is equal to half of its major axis.
8. Find equation of the circle passing through focus of parabola $y^2 + 8x = 0$ and foci of ellipse $25x^2 + 16y^2 = 400$.
9. Find the length of, and the equations to, the focal radii drawn to a point $\left(4\sqrt{3}, 5\right)$ of the ellipse $25x^2 + 16y^2 = 1600$.
10. Find equation of ellipse with centre at $(0, 1)$ and major axis parallel to y-axis. Also, it passes through $(2, 1)$ and $(0, 4)$.

9.8 Equations of Tangent and Normal of an Ellipse

As tangent and normal are very important to solve many physical problems, so we will discuss concept, conditions and equations of tangents and normals to an ellipse in this section.

9.8.1 Recognize tangent and normal to an ellipse

Like any other curve, tangent to an ellipse is the line which touches the ellipse at a certain point P. The point P is called point of tangency. Normal to the ellipse is a line which is perpendicular to the tangent at the point of tangency.

In the figure 9.44, l_1 is tangent to the ellipse at P and l_2 is the normal.

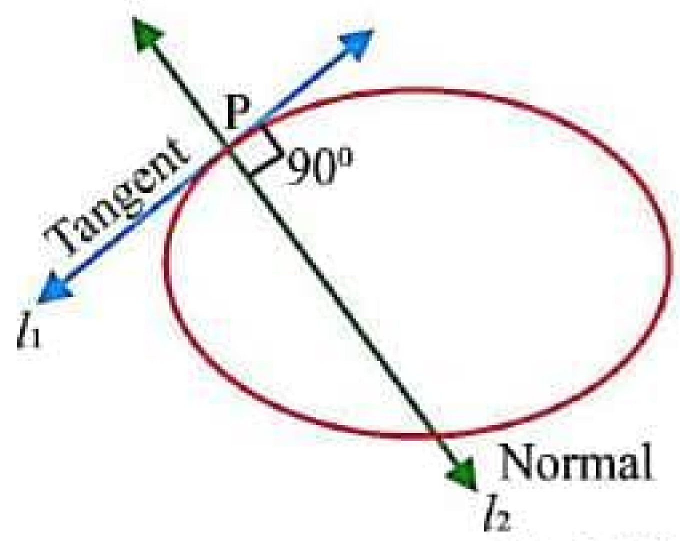


Fig 9.44

9.8.2 Find points of intersection of an ellipse with a line including the condition of tangency

As a matter of fact, a line can cut or touch an ellipse and sometimes it neither cuts nor touches the ellipse as shown in the figure 9.45. We will discuss the method of finding points of intersection of an ellipse with a line along with condition of tangency in this section.

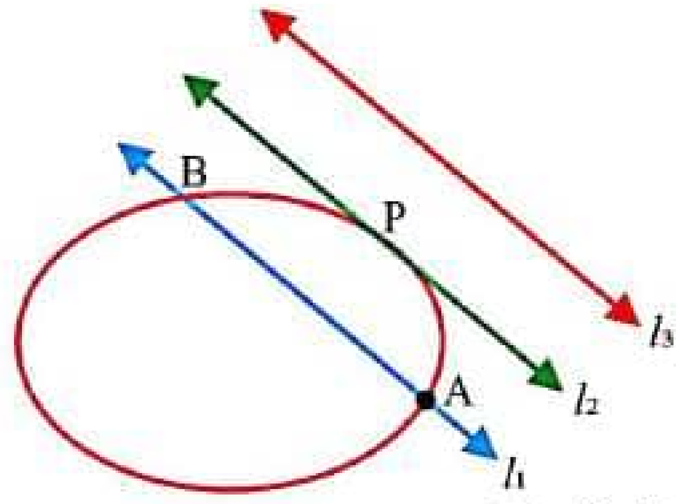


Fig 9.45

Consider a line $y = mx + c$ ...(i)

and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(ii)

Solving both equations simultaneously,

we get $b^2x^2 + a^2(mx + c)^2 = a^2b^2$

$\Rightarrow$ $(b^2 + a^2m^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$ ...(iii)

$\because$ Roots of quadratic equation (iii)

represent abscissas of the points of intersection.

$\therefore$ Abscissas of points of intersection will be the roots of (iii) and the corresponding ordinates of points of intersection will be obtained by

Substituting the values of x in (i)

Moreover, the nature of roots of quadratic equation (iii) will represent the nature of parallel lines, l_1, l_2 and l_3 each having slope m with respect to given ellipse.

Here, discriminant of equation (iii) is

$\Delta = 4m^2c^2a^4 - 4a^2(b^2 + a^2m^2)(c^2 - b^2)$

$= 4m^2c^2a^4 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4m^2b^2$

$= 4a^2b^2(b^2 - c^2 + a^2m^2)$

If $\Delta > 0$ then line will cut the ellipse because there will be two distinct real roots.

If $\Delta < 0$ then the line will neither cut nor touch the ellipse because there will be no real root.

If $\Delta = 0$ then line will be tangent to the ellipse because there will be only one real root.

i.e., $4a^2b^2(b^2 - c^2 + a^2m^2) = 0$

$\Rightarrow$ $b^2 - c^2 + a^2m^2 = 0$

$\Rightarrow$ $c^2 = b^2 + a^2m^2$

This is the condition of tangency of line $y = mx + c$ with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example: Show that the line $y = 2x + 4$ is tangent to the ellipse $4x^2 + 3y^2 = 12$. Also find point of contact.

Solution: We have

Line $y = 2x + 4$ ...(i)

and ellipse: $4x^2 + 3y^2 = 1$ ...(ii)

Solving equation (i) and (ii) simultaneously, we get $\quad 4x^2 + 3(2x+4)^2 - 12 = 0$

$\Rightarrow \quad 4x^2 + 3(4x^2 + 16x + 16) - 12 = 0$

$\Rightarrow \quad 16x^2 + 48x + 36 = 0$

$\Rightarrow \quad 4x^2 + 12x + 9 = 0 \qquad \ldots(iii)$

Here $\quad \Delta = 144 - 4(4)(9)$

$\quad = 144 - 144 = 0$

$\because \quad \Delta = 0$

$\therefore$ Given line is tangent to the given ellipse.

From equation (iii), we have $x = -\frac{b}{2a}$

i.e., $\quad x = -\frac{12}{8} = -\frac{3}{2}$

By using $x = -\frac{3}{2}$ in equation (i)

we get $\quad y = 2\left(-\frac{3}{2}\right) + 4$

$\quad = 1$

So, the point of intersection is $\left(-\frac{3}{2}, 1\right)$.

9.8.3 Find the equation of a tangent in slope form

Let $y = mx + c \qquad \ldots(i)$

be the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

By the condition of tangency

$\quad c^2 = b^2 + a^2m^2$

or $\quad c = \sqrt{b^2 + a^2m^2}$

By using this value of c in equation (i)

we get $\quad y = mx + \sqrt{b^2 + a^2m^2}$

This is the required equation of tangent in slope form to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example: Find the equation of tangent to $\frac{x^2}{16} + \frac{y^2}{9} = 1$ whose slope $= 2$.

Solution: Here slope $= m = 2, a^2 = 16$ and $b^2 = 9$.

We know that the equation of tangent in slope form is

$$y = mx + \sqrt{b^2 + a^2m^2}$$

i.e., $\quad y = 2x + \sqrt{9 + 16 \times 4}$

$\quad y = 2x + \sqrt{73}$

This is the required tangent.

9.8.4 Find the equation of a tangent and a normal to an ellipse at a point

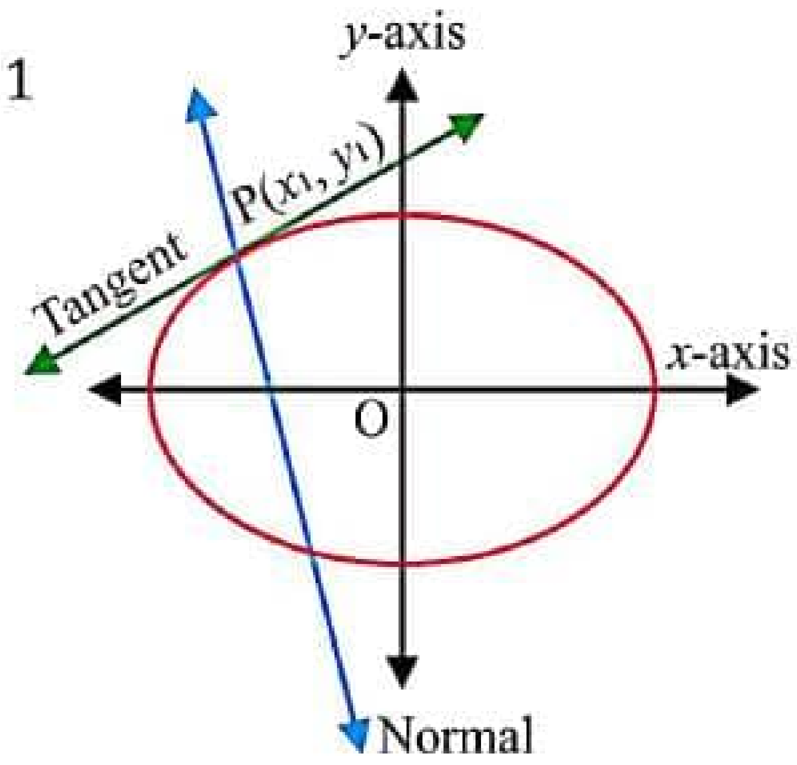


Fig 9.46

Let $P(x_1, y_1)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiating w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{b^2x}{a^2y}}$$

Now, slope of tangent at $(x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$= -\frac{b^2x_1}{a^2y_1}$$

By point-slope form, the equation of tangent will be

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\Rightarrow \frac{x_1(x - x_1)}{a^2} + \frac{y_1(y - y_1)}{b^2} = 0$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = 0$$

$$\Rightarrow \boxed{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

This is the required equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

$\because$ Normal is perpendicular to the tangent at $P(x_1, y_1)$

$\therefore$ slope of normal $= \frac{a^2y_1}{b^2x_1}$ at (x_1, y_1).

By point-slope form, the equation of normal will be

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

$$\Rightarrow \frac{b^2(y - y_1)}{y_1} = \frac{a^2(x - x_1)}{x_1}$$

$$\Rightarrow \frac{a^2x}{x_1} - \frac{b^2y}{y_1} - a^2 + b^2 = 0$$

or $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

This is the equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Example: Find equation of tangent and normal to $\frac{x^2}{5}+\frac{y^2}{3}=1$ at $\left(\sqrt{\frac{5}{2}},\sqrt{\frac{3}{2}}\right)$.

Solution: We have ellipse: $\frac{x^2}{5}+\frac{y^2}{3}=1$

differentiating w.r.t x

$$\frac{2x}{5}+\frac{2y}{3}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2x}{5}\times\frac{3}{2y}$$

$$\frac{dy}{dx}=-\frac{3x}{5y}$$

Now, slope of tangent at $\left(\sqrt{\frac{5}{2}},\sqrt{\frac{3}{2}}\right)=m=\left(\frac{dy}{dx}\right)_{\left(\sqrt{\frac{5}{2}},\sqrt{\frac{3}{2}}\right)}$

$$=-\sqrt{\frac{3}{5}}$$

So, by point-slope form, the equation of tangent will be

$$y-y_1=m(x-x_1)$$

i.e., $\quad y-\sqrt{\frac{3}{2}}=-\sqrt{\frac{3}{5}}\left(x-\sqrt{\frac{5}{2}}\right)$

$\because$ Normal is perpendicular to the tangent at $\left(\sqrt{\frac{5}{2}},\sqrt{\frac{3}{2}}\right)$

$\therefore$ slope of normal $=\sqrt{\frac{5}{3}}=m'$

By point-slope form, the equation of normal will be

$$y-y_1=m'(x-x_1)$$

i.e., $y-\sqrt{\frac{3}{2}}=\sqrt{\frac{5}{3}}\left(x-\sqrt{\frac{5}{2}}\right)$

Exercise 9.4

1. Find the condition when line $y=\sqrt{5}x+c$ is tangent to the ellipse $4x^2+9y^2=36$.
2. Show that the line $x=2y+4$ touches the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$. Also find point of contact.

3. Find the condition of tangency of line $y = mx + c$ to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

4. Find the condition of tangency of given line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   (i) $\frac{x}{p} + \frac{y}{q} = 1$ (ii) $x\cos\alpha + y\sin\alpha = p$

   (iii) $lx + my + n = 0$

5. Find the equation of tangent to $\frac{x^2}{5} + \frac{y^2}{4} = 1$ with slope 3.

6. Find the equation of tangent and normal to

   (i) $9x^2 + 25y^2 = 225$ at $\left(3, \frac{12}{5}\right)$

   (ii) $49x^2 + 64y^2 = 64 \times 49$ at $(8\cos\alpha, 7\sin\alpha)$

7. Find the equation of tangent and normal at the ends of the latus rectum with positive abscissa of the ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$.

8. Find the equation of tangent to the ellipse $\frac{x^2}{5} + \frac{9y^2}{20} = 1$ at the points where abscissa is 1.

9.9 Hyperbola

In previous chapter we defined hyperbola as conic section of right circular cone but in this section, we will discuss hyperbola in detail on the basis of eccentricity and directrix.

9.9.1 Define hyperbola

Hyperbola is defined on the basis of two geometrical properties, one is directrix-focus property whereas the other one is based on the distances of a point from two fixed points.

Definition 1: A hyperbola is the set of all the points in a plane whose distance from a fixed point bears a constant ratio to its distance from a fixed line such that the ratio is greater than 1.

The fixed point, fixed line and ratio are called focus, directrix and eccentricity respectively. Hyperbola has two foci and two directrices as shown in the figure 9.47.

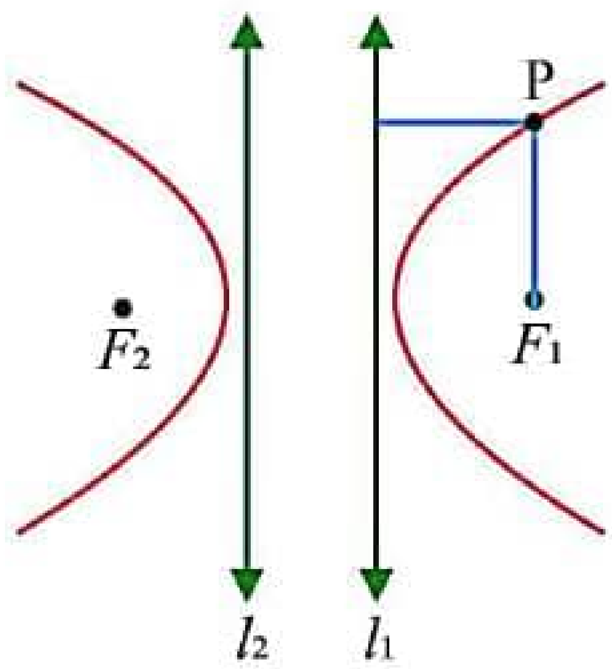


Fig 9.47

Definition 2: The locus of a point, the difference of whose distances from two fixed points is constant, is called hyperbola. The fixed points are foci.

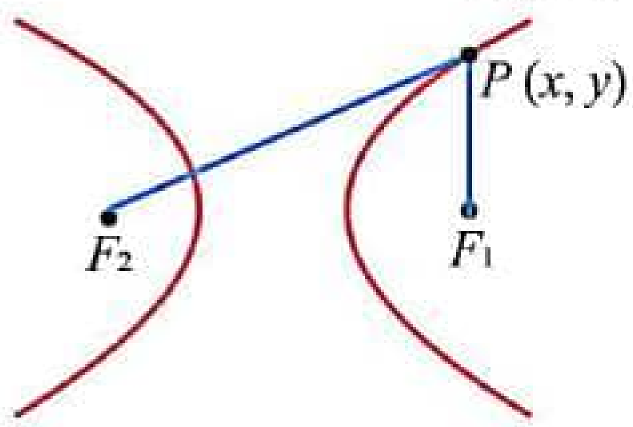


Fig 9.48

Let $P(x, y)$ be any point of hyperbola whereas F_1 and F_2 are two foci as shown in the figure 9.48 then

$|\overline{PF_2}| - |\overline{PF_1}| =$ constant (say k).

9.9.2 Define elements of hyperbola (i.e., centre, foci, vertices, directrices, transverse and conjugate axes, eccentricity, focal chord and latera recta)

As discussed above hyperbola has two fixed points and two fixed lines which are called foci and directrices respectively. In the figure 9.49, F_1, F_2 are foci whereas l_1, l_2 are directrices of hyperbola.

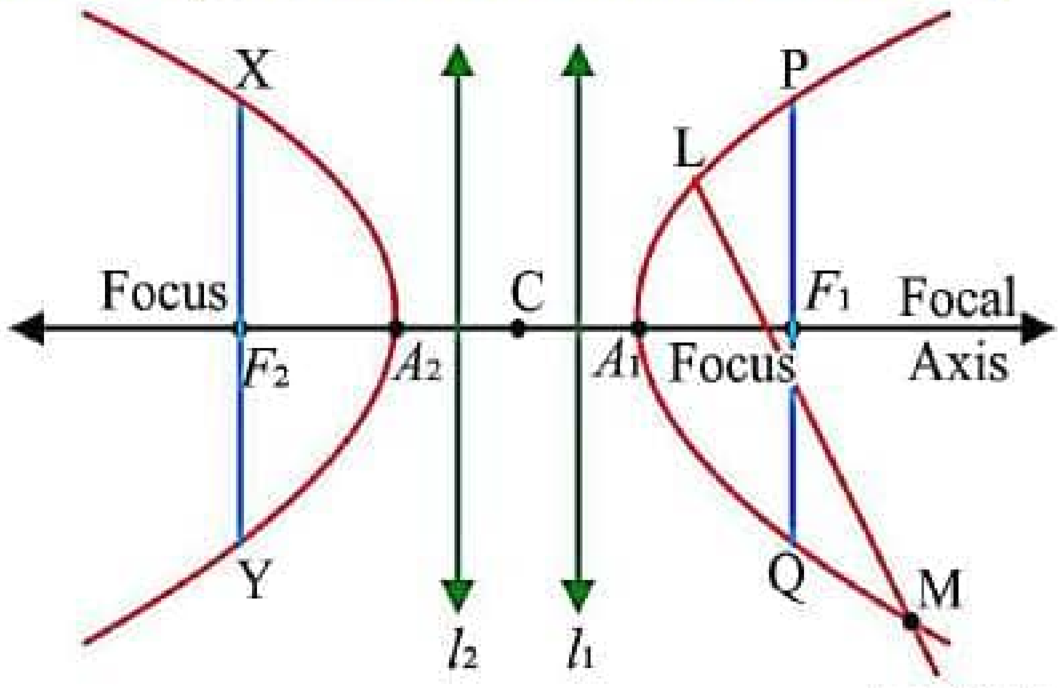


Fig 9.49

The mid-point of the line segment joining the foci is called centre. In the figure point C is the centre of hyperbola. The line through foci is called focal axis.

Hyperbola intersects focal axis at two points called vertices. In the Fig. 9.49 A_1 and A_2 are two vertices of the hyperbola. The two parts of hyperbola are called its branches. Any chord of hyperbola through any one of its foci is called focal chord. In the figure $\overline{LM}$ is a focal chord. The focal chord perpendicular to the focal axis is called latus rectum. In the figure $\overline{PQ}$ and $\overline{XY}$ are the latera recta (plural of latus rectum) of the hyperbola. The ratio of distances of any point from focus and directrix is called eccentricity and is denoted by e where $e > 1$.

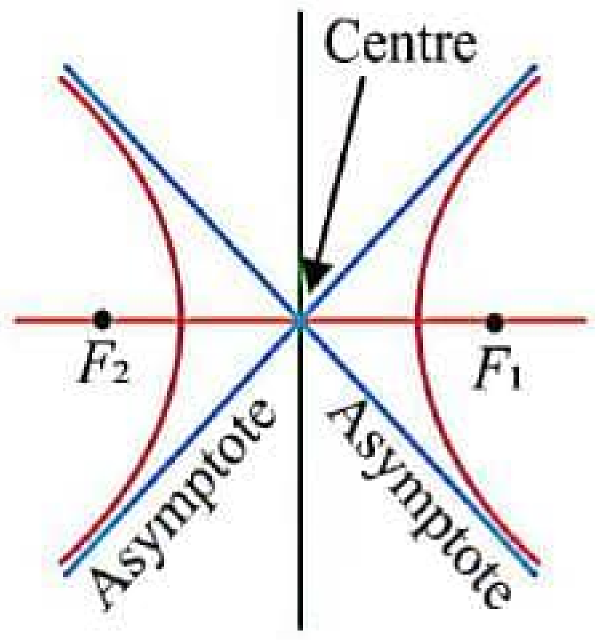


Fig 9.50

A line passing through centre and gets closer and closer to hyperbola but never touches it is called asymptote. There are two asymptotes of any hyperbola as shown in the figure 9.50.

Let distance of focus and vertex from centre are c and a respectively. Asymptotes are in fact diagonal lines of a rectangle which extends a unit from centre on either side on focal axis and it extends b units from centre on either side on the line perpendicular to focal axis and through centre as shown in the Fig. 9.51 where we define b as $b = \sqrt{c^2 - a^2}$.

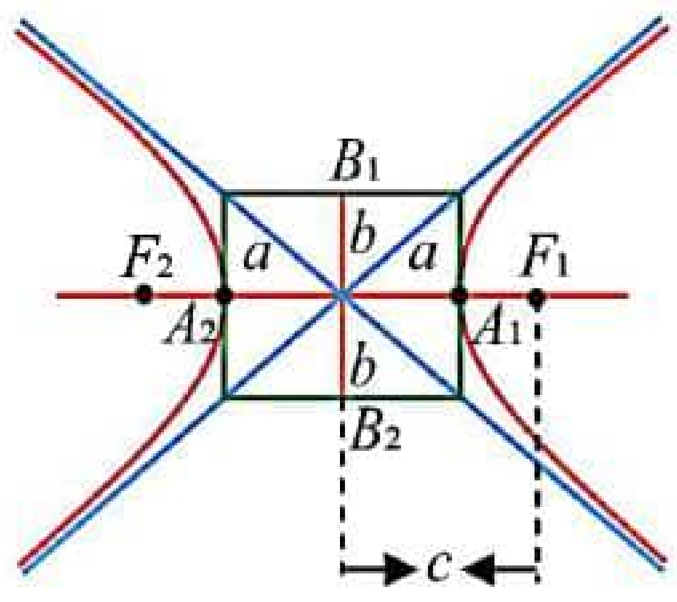


Fig 9.51

The line segments A_1A_2 and B_1B_2 are called transverse and conjugate axes respectively where A_1 and A_2 are the vertices but B_1 and B_2 are the ends of conjugate axis of hyperbola.

The relationship of a, b and c is pictured geometrically in the figure.

Relation is: $c^2 = a^2 + b^2$

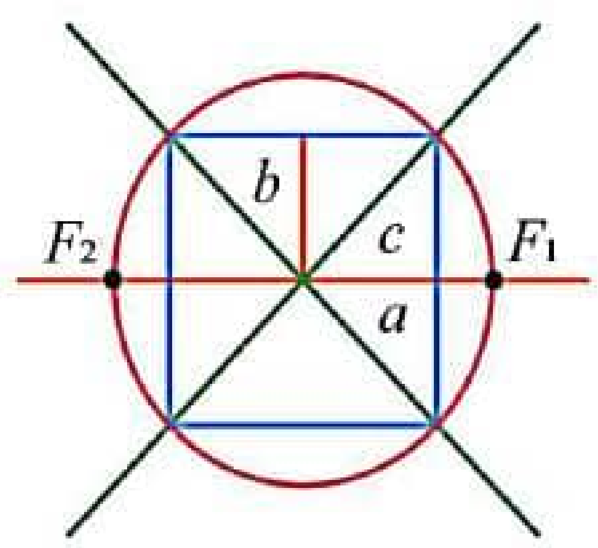


Fig 9.52

Property: The differences of the focal distances of a point on a hyperbola is equal to the length of its transverse axis as explained with the help of the figure 9.53. We take vertex A as a point of hyperbola.

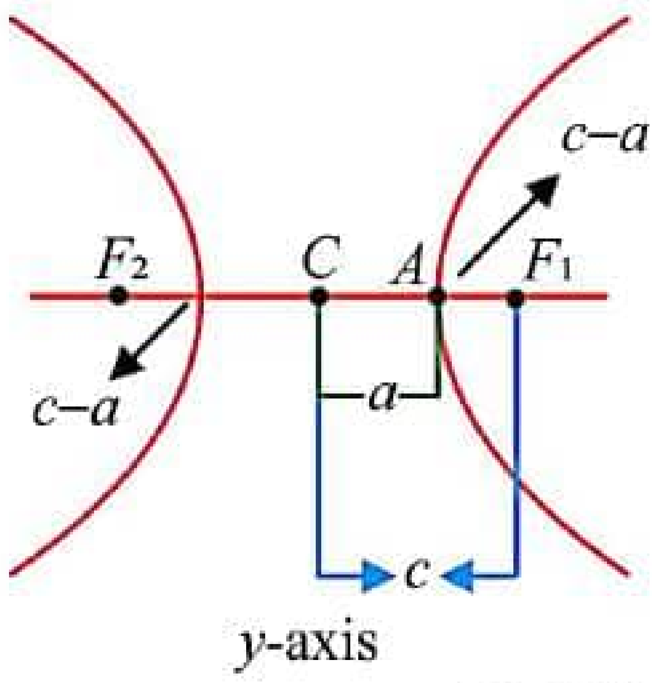


Fig 9.53

By definition 2 of hyperbola

$$m\overline{AF_2} - m\overline{AF_1} = k \text{ (constant)}$$

i.e., $\{(c-a)+2a\}-(c-a) = k$

$\Rightarrow$ $c - a + 2a - c + a = k$

or $k = 2a.$

This is valid for any point of hyperbola.

Hence difference of focal distances of a point on a hyperbola is equal to the length of its transverse axis.

Distance of directrix from the centre and the relation $c = ae$

Consider a hyperbola in which distance of a focus and vertex from centre is c and a units respectively and l is the directrix cutting focal axis at point D as shown in the figure 9.54.

Now, $\dfrac{m\overline{F_1A_1}}{m\overline{A_1D}} = e > 1 \Rightarrow m\overline{F_1A_1} = e(m\overline{A_1D})$ as point A_1 is nearer to D then F_1 and

$$\frac{m\overline{F_1A_2}}{m\overline{A_2D}} = -e \text{ (as } m\overline{F_1A_2} = -m\overline{A_2F_1})$$

$\Rightarrow$ $m\overline{F_1A_2} = -e(m\overline{A_2D})$

$\Rightarrow$ $m\overline{A_2F_1} = e(m\overline{A_2D})$

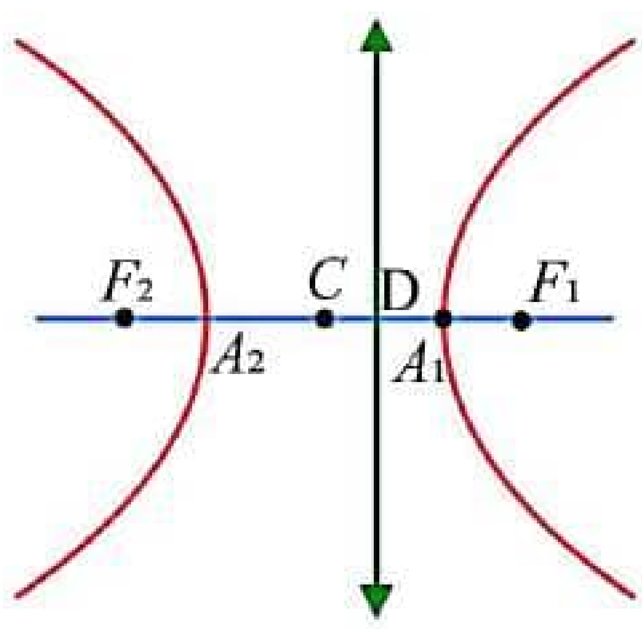


Fig 9.54

The points A_1 and A_2 are on the opposite sides of l. Take C as origin and $\overline{CF_1}$ as positive x-axis. Now, by definition of hyperbola

$$\begin{aligned} m\overline{A_1A_2} &= m\overline{A_2F_1} - m\overline{A_1F_1} \\ &= m\overline{A_2F_1} - m\overline{F_1A_1} \\ &= e(m\overline{A_2D} + m\overline{A_1D}) \\ &= e(m\overline{A_2C} + m\overline{CD}) + e(m\overline{A_1C} + m\overline{CD}) \end{aligned}$$

$$2a = 2e(m\overline{CD}) \qquad (\because m\overline{A_2C} = -m\overline{A_1C})$$

$$\Rightarrow \quad m\overline{CD} = \frac{a}{e} \qquad \ldots\text{(i)}$$

Hence, directrix l is at a distance of $\dfrac{a}{e}$ from the centre.

Similarly, $m\overline{CF_1} = m\overline{CA_1} + m\overline{A_1F_1}$

$$= m\overline{CA_1} + e(m\overline{A_1D})$$

$$c = a + e(m\overline{CA_1} - m\overline{CD})$$

$$\Rightarrow \quad c = a + e\left(a - \frac{a}{e}\right)$$

$$\Rightarrow \quad c = a + ae - a$$

or $$c = ae \qquad \ldots(ii)$$

9.10 Standard Form of Equation of Hyperbola

The simplest form of equation of hyperbola is when the coordinate axes are positioned in such a way that centre of hyperbola is at the origin and the transverse axis and conjugate axis are on the coordinate axes. The two possible such orientation are shown in the figure 9.55 and 9.56.

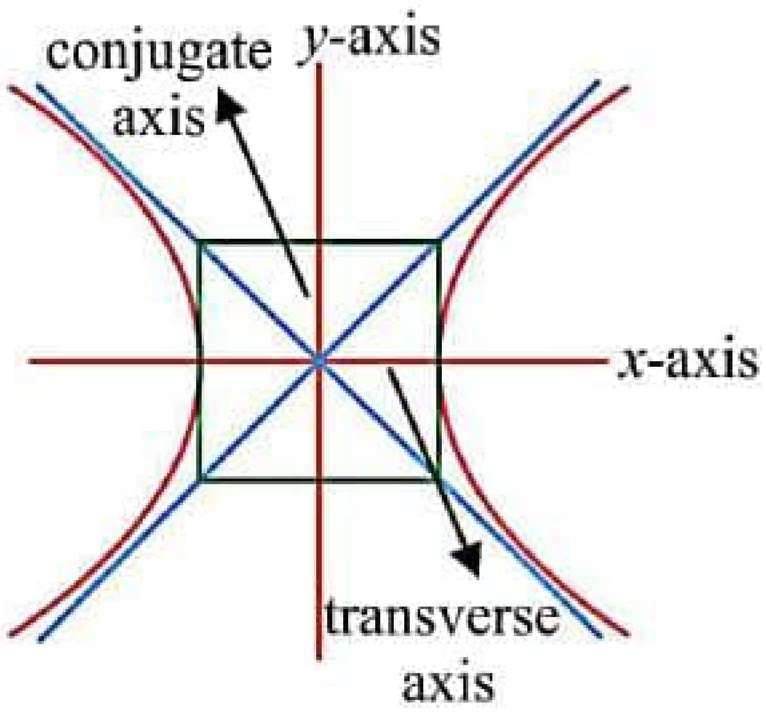


Fig 9.55

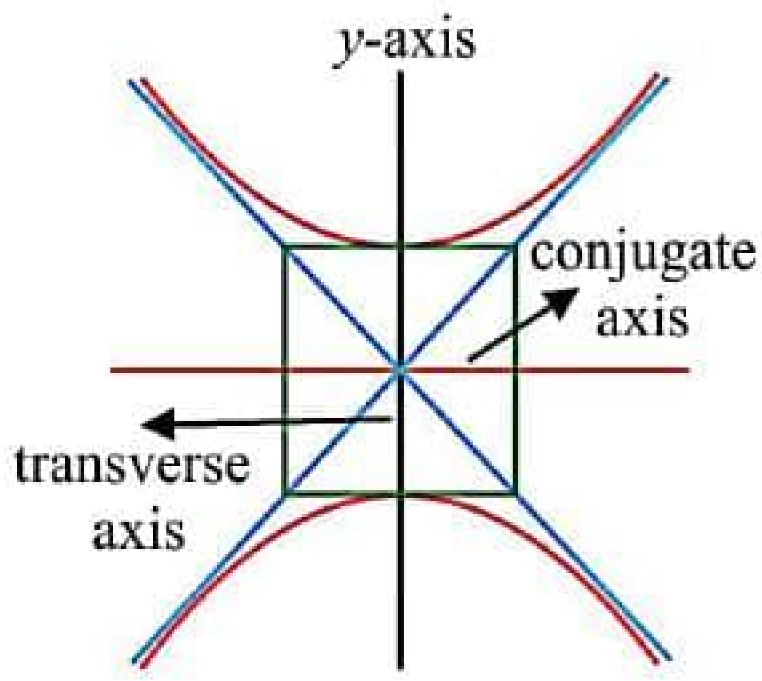


Fig 9.56

These are the standard positions of hyperbola and the resulting equations are called standard equations which will be derived in the next section.

9.10.1 Derive the standard form of equation of a hyperbola and identify its elements

(a) Standard equation of hyperbola when transverse axis is along *x*-axis.

Let $P(x, y)$ be any point on hyperbola with centre at origin, transverse axis on *x*-axis and conjugate axis along y-axis as shown in the figure 9.57 whereas foci are $F_1(c, 0)$ and $F_2(-c, 0)$. The length of transverse axis is $2a$.

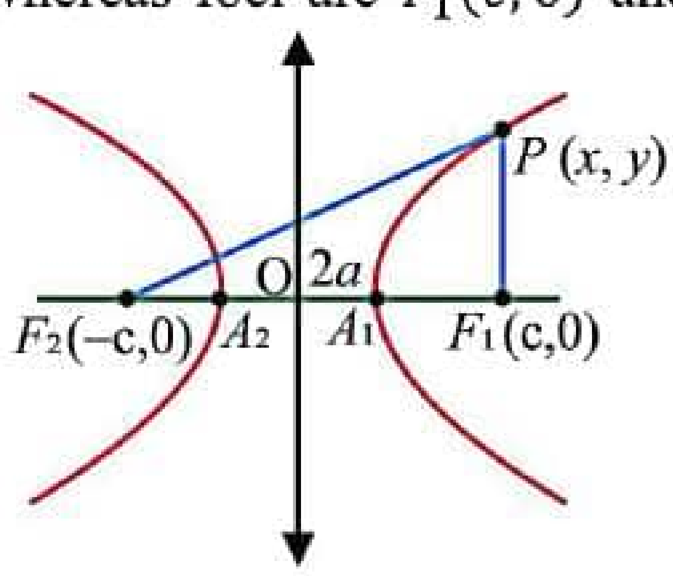


Fig 9.57

By the definition of hyperbola

$$|\overline{PF_2}| - |\overline{PF_1}| = 2a$$

i.e., $$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

or $$\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Squaring both sides

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$\Rightarrow \quad 4cx - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \quad \frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

Again, squaring both sides

$$\frac{cx^2}{a^2} - 2cx + a^2 = x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow \quad c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$\Rightarrow \quad (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

Dividing both sides by $(c^2 - a^2)a^2$

we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

i.e., $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (\because c^2 = a^2 + b^2)$$

This is the required equation of hyperbola whose transverse axis is along x-axis.

with centre at origin whereas foci are $(0, \pm c)$, vertices are $(0, \pm a)$, ends of conjugate axis are $(\pm bi, 0)$, equations of directrices are $y = \pm\frac{a}{e}$ and latus rectum $= \frac{2b^2}{a}$.

With centre at origin whereas foci are $(\pm c, 0)$, vertices are $(\pm a, 0)$.

In order to find ends of conjugate axis, we find y-intercepts of hyperbola by using $x = 0$ in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we get $\quad y^2 = -b^2$

$$\Rightarrow \quad y = \pm bi$$

So, the ends of conjugate axis are $(0, \pm bi)$

$\because$ Distance of directrix from centre is $\frac{a}{e}$ as we studied in section 9.9.2.

$\therefore$ Equation of directrices will be $x = \pm\frac{a}{e}$.

Length of latus rectum of hyperbola

Let PQ be the latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

with a focus $F_1(c, 0)$ as shown in Fig. 9.58

$\because$ latus rectum PQ passes through focus $F_1(c, 0)$ and perpendicular to the focal axis.

$\therefore$ its equation will be $x = c$ ...(ii)

Solving (i) and (ii) simultaneously

We get $$\frac{c^2}{a^2} - \frac{y^2}{b^2} = 1$$

i.e., $$y = \pm b\frac{\sqrt{c^2 - a^2}}{a}$$

$$y = \pm\frac{b^2}{a} \qquad (\because c^2 - a^2 = b^2)$$

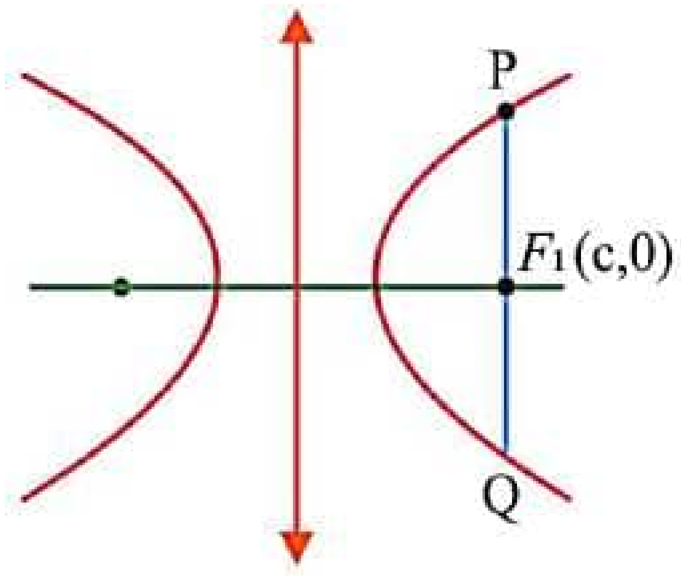


Fig 9.58

So, the coordinates of P and Q are $\left(c, \frac{b^2}{a}\right)$ and $\left(c, \frac{-b^2}{a}\right)$

Now, $$|\overline{PQ}| = \sqrt{\left(\frac{2b^2}{a}\right)^2} = \frac{2b^2}{a}$$

So, the length of latus rectum is $\frac{2b^2}{a}$.

(b) Standard equation of hyperbola when transverse axis is along y-axis

Let $P(x, y)$ be any point on hyperbola with centre at origin, transverse axis along y-axis and conjugate axis along x-axis as shown in the figure 9.59, whereas foci are $F_1(0, c)$ and $F_2(0, -c)$. The length of transverse axis is $2a$.

By the definition of hyperbola

$$|\overline{PF_2}| - |\overline{PF_1}| = 2a$$

i.e., $$\sqrt{x^2 + (y+c)^2} - \sqrt{x^2 + (y-c)^2} = 2a$$

or $$\sqrt{x^2 + (y+c)^2} = 2a + \sqrt{x^2 + (y-c)^2}$$

Squaring both sides

$$x^2 + (y+c)^2 = 4a^2 + 4a\sqrt{x^2 + (y-c)^2} + x^2 + (y-c)^2$$

$$\Rightarrow \quad 4cy - 4a^2 = 4a\sqrt{x^2 + (y-c)^2}$$

$$\Rightarrow \quad \frac{cy}{a} - a = \sqrt{x^2 + (y-c)^2}$$

Again, squaring both sides

$$\frac{c^2y^2}{a^2} - 2cy + a^2 = x^2 + y^2 - 2cy + c^2$$

$$\Rightarrow \quad c^2y^2 + a^4 = a^2x^2 + a^2y^2 + a^2c^2$$

$$\Rightarrow \quad (c^2 - a^2)y^2 - a^2x^2 = a^2(c^2 - a^2)$$

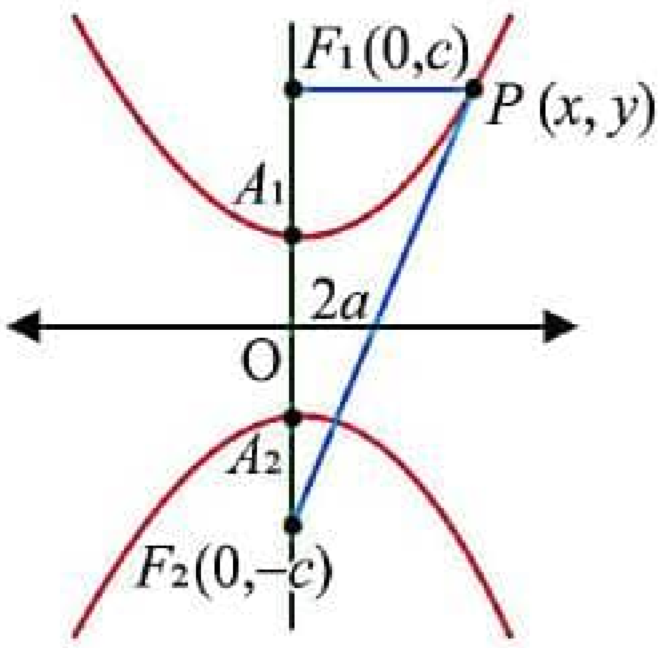


Fig 9.59

Dividing both sides by $a^2(c^2 - a^2)$

we get

$$\frac{y^2}{a^2} - \frac{x^2}{c^2 - a^2} = 1$$

i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $\qquad (\because c^2 = a^2 + b^2)$

This is the required equation of hyperbola whose transverse axis is along y-axis.

Equilateral or Rectangular hyperbola

A hyperbola, in which transverse axis and conjugate axis are of same length is called rectangular or equilateral hyperbola.

i.e., $2a = 2b$

i.e., $b = a$

So, the equation of hyperbola is: $x^2 - y^2 = a^2$

where $e = \sqrt{2}$

because $c^2 = a^2 + b^2$

i.e., $a^2e^2 = 2a^2$ $\qquad (\because c = ae)$

$\Rightarrow$ $e = \sqrt{2}$

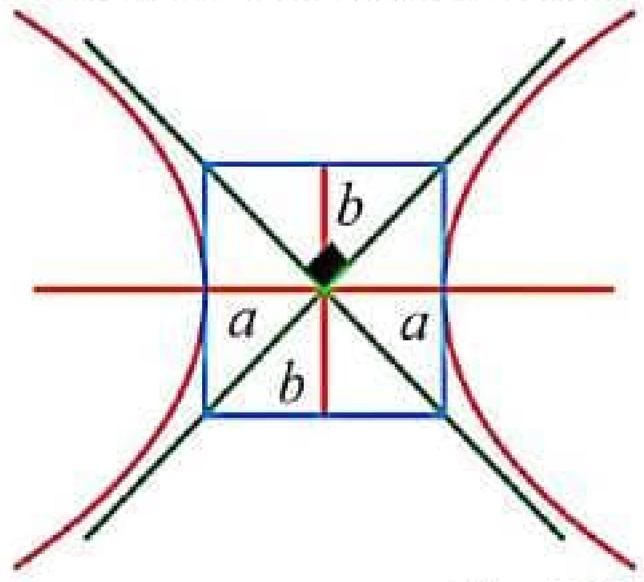


Fig 9.60

In rectangular hyperbola asymptotes are perpendicular to each other as shown in the figure 9.60.

Conjugate Hyperbola

The conjugate hyperbola of a given hyperbola is the hyperbola whose transverse and conjugate axes are respectively conjugate and transverse axes of given hyperbola.

Thus, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Standard forms of Translated Hyperbolas

If centre of hyperbola is not at origin but the transverse and conjugate axes are parallel to the coordinate axes then it is standard form of translated hyperbola and its equations along with its elements are given below.

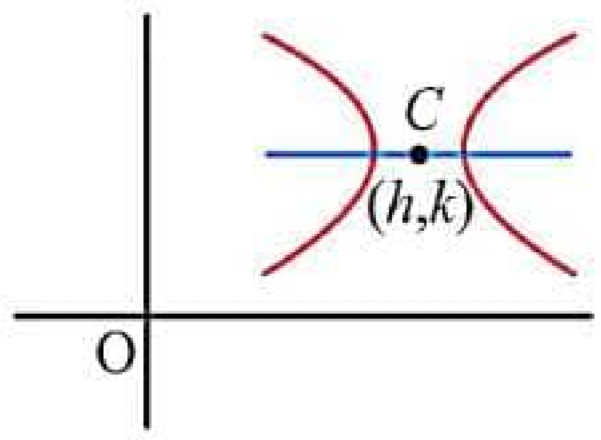


Fig 9.61

(a) Equation of hyperbola when centre is at (h, k) and transverse axis parallel to x-axis is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Here foci are $(h \pm c, k)$, vertices are $(h \pm a, k)$, ends of conjugate axes are $(h, k \pm bi)$.

Equations of directrices are: $x - h = \pm\frac{a}{e}$ and latus rectum is $\frac{2b^2}{a}$.

(b) Equation of hyperbola when centre is at (h, k) and transverse axis parallel to y-axis is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Here foci are $(h, k \pm c)$, vertices are $(h, k \pm a)$, ends of conjugate axes are $(h \pm bi, k)$. Equations of directrices are $y-k=\pm\frac{a}{e}$ and latus rectum $=\frac{2b^2}{a}$.

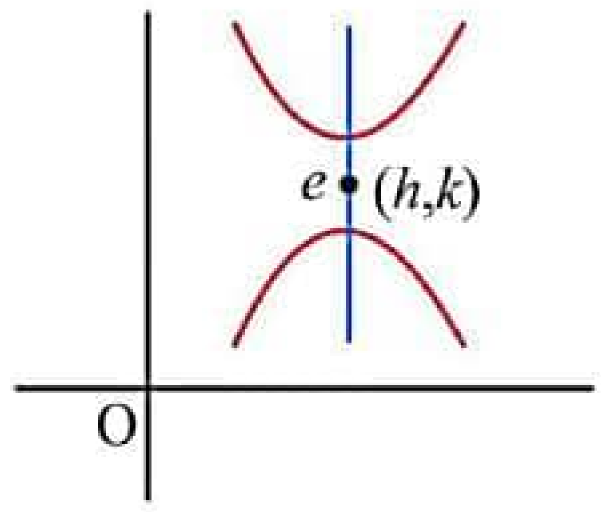


Fig 9.62

(c) **General equation of hyperbola when transverse and conjugate axes are parallel to coordinate axes.**

Consider a hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

On simplification, it becomes

$$b^2(x^2-2hx+h^2)-a^2(y^2-2ky+k^2)=a^2b^2$$

$$\Rightarrow \quad b^2x^2-2hb^2x+b^2h^2-a^2y^2+2ka^2y-a^2k^2-a^2b^2=0$$

$$\Rightarrow \quad b^2x^2-a^2y^2-2hb^2x+2ka^2y+b^2h^2-a^2k^2-a^2b^2=0 \qquad \ldots(i)$$

Let $A=b^2, B=-a^2, G=-2hb^2, F=2ka^2$ and $C=b^2h^2-a^2k^2-a^2b^2$

So, equation (i) becomes

$$Ax^2+By^2+Gx+Fy+C=0 \qquad \ldots(ii)$$

Where A and B are non-zero and have different signs.

Equation (ii) is the general equation of hyperbola.

Summary of standard equations of hyperbola and related terms

Equation	Related Terms
(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Centre at origin and transverse axis is along *x*-axis.	Foci are $(\pm c, 0)$ Vertices are $(\pm a, 0)$ Ends of conjugate axis are $(0, \pm bi)$ Directrices: $x=\pm\frac{a}{e}$ Asymptotes: $y=\pm\frac{b}{a}x$
(ii) $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ Centre at origin and transverse axis is along *y*-axis.	Foci are $(0, \pm c)$ Vertices are $(0, \pm a)$ Ends of conjugate axis are $(\pm bi, 0)$ Directrices: $y=\pm\frac{a}{e}$ Asymptote: $y=\pm\frac{a}{b}x$

Equation	Related Terms
(iii) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ Centre at (h,k) and transverse axis is parallel to x-axis.	Foci are $(h\pm c,k)$ Vertices are $(h\pm a,k)$ Ends of conjugate axis are $(h,k\pm bi)$ Directrices: $x-h=\pm\frac{a}{e}$ Asymptote: $y-k=\pm\frac{b}{a}(x-h)$
(iv) $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ Centre at (h,k) and transverse axis is parallel to y-axis.	Foci: $(h,k\pm c)$ Vertices: $(h,k\pm a)$ Ends of conjugate axis: $(h,\pm bi,k)$ Directrices: $y-k=\pm\frac{a}{e}$ Asymptote: $y-k=\pm\frac{a}{b}(x-h)$

Note: For all standard equations of hyperbola, we have

Latus rectum $=\frac{2b^2}{a}$

$c=ae$ and $c^2=a^2+b^2$

Similarities and differences between ellipse and hyperbola

- Similarities for standard forms along x-axis

Ellipse		Hyperbola	
(i)	Two foci: $(\pm c,0)$	(i)	Two foci: $(\pm c,0)$
(ii)	Two vertices: $(\pm a,0)$	(ii)	Two vertices: $(\pm a,0)$
(iii)	Two directrices: $x=\pm\frac{a}{e}$	(iii)	Two directrices: $x=\pm\frac{a}{e}$
(iv)	Length of latus rectum $=\frac{2b^2}{a}$	(iv)	Length of latus rectum $=\frac{2b^2}{a}$
(v)	$c=ae$	(v)	$c=ae$
(vi)	axes are $2a$ and $2b$	(vi)	axes are $2a$ and $2b$

- Differences

Ellipse		Hyperbola	
(i)	$a>c$	(i)	$a<c$
(ii)	$c^2=a^2-b^2$	(ii)	$c^2=a^2+b^2$
(iii)	Ellipse is closed curve	(iii)	Hyperbola is not closed figure
(iv)	If $b=a$ then it is an auxiliary circle with $e=0$ and foci coincide	(iv)	If $b=a$ then it is an rectangular hyperbola with $e=\sqrt{2}$ and foci do not coincide

Example 1. Find eccentricity, foci, vertices, ends of conjugate axis and latus rectum of hyperbola: $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution: We have hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Comparing with standard equation.

we get $a^2 = 16, b^2 = 9$, transverse axis is along x-axis and centre at origin.

we know that

$$c^2 = a^2 + b^2$$

i.e., $c^2 = 16 + 9 = 25$

So, $a = 4, b = 3$ and $c = 5$.

We obtain eccentricity by $c = ae$

i.e., $\frac{5}{4} = e$

Foci $= (\pm c, 0) = (\pm 5, 0)$

Vertices $(\pm a, 0) = (\pm 4, 0)$

Ends of conjugate axis $= (0, \pm bi) = (0, \pm 3i)$

and Latus rectum $= \frac{2b^2}{a}$

$$= \frac{18}{4} = \frac{9}{2}$$

Example 2. Find eccentricity, equation of directrices and equations of asymptotes of hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$.

Solution: Comparing given hyperbola with standard equation of hyperbola.

we get $a^2 = 9, b^2 = 4$, transverse axis along y-axis and centre at origin.

We know that

$$c^2 = a^2 + b^2$$

i.e., $c^2 = 9 + 4 = 13$

So, $a = 3, b = 2$ and $c = \sqrt{13}$.

We obtain eccentricity by $c = ae$

i.e., $\sqrt{13} = 3e$

$\Rightarrow$ $e = \frac{\sqrt{13}}{3}$

$\because$ Transverse axis is along y-axis

∴ Its directrices will be

$$y = \pm\frac{a}{e}$$

i.e., $$y = \pm\frac{3}{\frac{\sqrt{13}}{3}}$$

or $$y = \pm\frac{9}{\sqrt{13}}$$

Also, the equation of asymptotes will be

$$y = \pm\frac{a}{b}x$$

i.e., $$y = \pm\frac{3}{2}x.$$

Example 3. Find centre, foci, vertices, latus rectum and equations of directrices, for the hyperbola $\frac{(x-3)^2}{64} - \frac{(y+4)^2}{36} = 1$

Solution: Comparing given equation of hyperbola with

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

We get $h = 3$ and $k = -4$

Also, $a^2 = 64 \quad \Rightarrow \quad a = 8$

and $b^2 = 36 \quad \Rightarrow \quad b = 6$

Its transverse axis is parallel to *x*-axis with centre $= (h, k) = (3, -4)$

We know that

$$c^2 = a^2 + b^2$$

$\Rightarrow \quad c^2 = 64 + 36$

$\Rightarrow \quad c^2 = 100$

$\Rightarrow \quad c = 10$

Now, foci $= (h \pm c, k)$

$= (3 \pm 10, -4)$

So, foci are $(13, -4)$ and $(-7, -4)$

Vertices $= (h \pm a, k)$

$= (3 \pm 8, -4)$

So, vertices are $(11, -4)$ and $(-5, -4)$

Latus rectum $= \frac{2b^2}{a}$

$= \frac{2(36)}{8}$

$= 9$

Here, $c = ae$

i.e., $10 = 8e$

$\Rightarrow \quad e = \frac{5}{4}$

$\because$ Transverse axis is parallel to x-axis

$\therefore$ Equations of directrices will be

$$x - h = \pm\frac{a}{e}$$

i.e., $x - 3 = \pm\frac{8}{\frac{5}{4}}$

$\Rightarrow \quad x - 3 = \pm\frac{32}{5}$

9.10.2 Find the equation of hyperbola with the following given elements

- **transverse and conjugate axes with centre at origin,**
- **two points,**
- **eccentricity, latera recta and transverse axes,**
- **focus, eccentricity and centre,**
- **focus, centre and directrix.**

The equation of hyperbola can be found with different conditions and elements. Here we discuss some of them.

(a) When transverse and conjugate axes are given with centre at origin

The method is explained with the help of the following example.

Example: Find the equation of hyperbola if transverse axis and conjugate axis are 8 and 6 units long respectively, where centre is at origin and transverse axis is along y-axis.

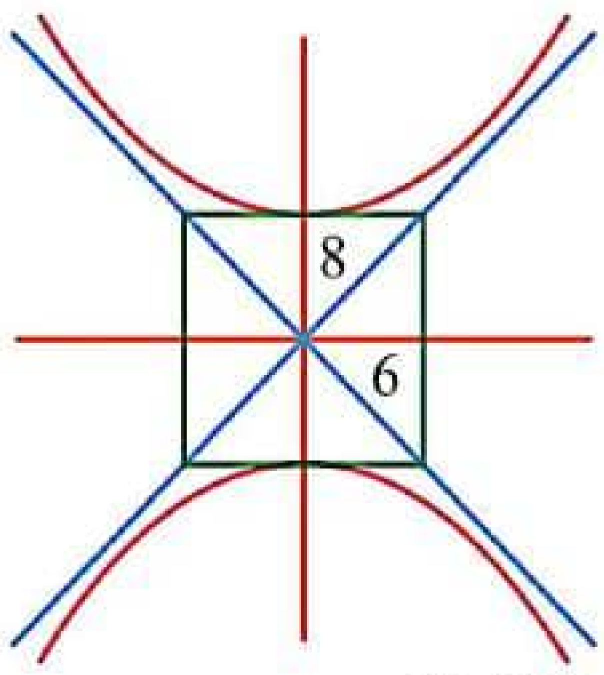


Fig 9.63

Solution: Here

$2a = 8$ and $2b = 6$

$\Rightarrow a = 4 \quad \Rightarrow b = 3$

$\because$ centre is at origin and transverse axis is along y-axis.

$\therefore$ Its equation will be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

i.e., $\frac{y^2}{16} - \frac{x^2}{9} = 1 \Rightarrow 9y^2 - 16x^2 = 144$

(b) When two points of hyperbola are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola with centre at origin and transverse axis along y-axis, such that the hyperbola passes through the points $\left(\frac{1}{2},\frac{\sqrt{5}}{2}\right)$ and $\left(\frac{1}{\sqrt{8}},\frac{-3}{\sqrt{8}}\right)$.

Solution: Centre is at origin and transverse axis is along y-axis.

∴ its equation will be $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ ...(i)

∵ $\left(\frac{1}{2},\frac{\sqrt{5}}{2}\right)$ lies on the hyperbola

∴ we have

$$\frac{5}{4a^2}-\frac{1}{4b^2}=1 \quad \text{...(ii)}$$

∵ $\left(\frac{1}{\sqrt{8}},\frac{-3}{\sqrt{8}}\right)$ is on hyperbola

∴ we have

$$\frac{9}{8a^2}-\frac{1}{8b^2}=1 \quad \text{...(iii)}$$

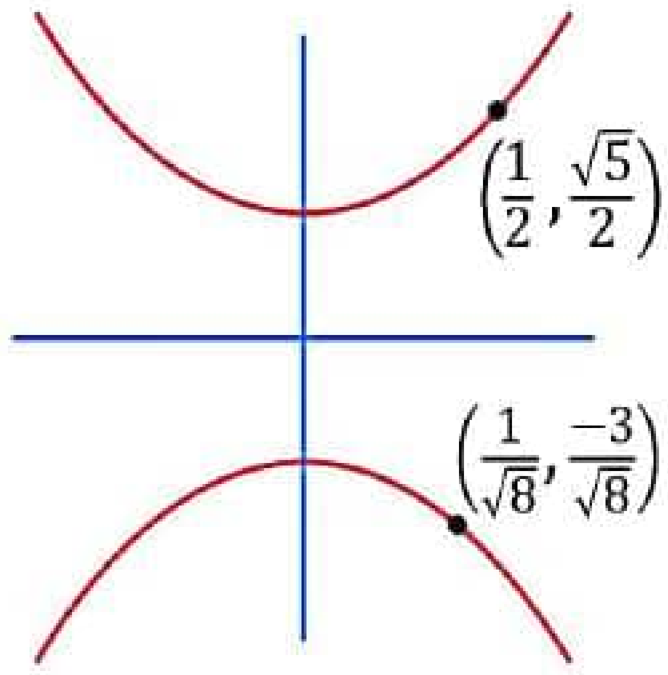


Fig 9.64

Multiplying equation (ii) by $\frac{1}{2}$ and subtracting resultant equation from (iii)

we get

$$\frac{4}{8a^2}=\frac{1}{2}$$

⇒ $a^2=1$

By using $a^2=1$ in equation (ii)

We get $\frac{5}{4}-\frac{1}{4b^2}=1$

⇒ $\frac{1}{4}=\frac{1}{4b^2}$ ⇒ $b^2=1$

By using values of a^2 and b^2 in (i)

We get $y^2-x^2=1$

(c) When eccentricity, latera recta or transverse axis are given

The method is explained with the help of following examples.

Example 1. Find the equation of hyperbola when centre is at origin and transverse axis is along x-axis with the length 10 units, whereas eccentricity is $\sqrt{3}$.

Solution: ∵ Centre is at origin and transverse axis is along x-axis

∴ Its equation will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

Here, $e = \sqrt{3}$

and $2a = 10$

i.e., $a = 5 \Rightarrow a^2 = 25$

Now, $c = ae$

$\Rightarrow c = 5\sqrt{3}$

We know that

$c^2 = a^2 + b^2$

i.e., $75 = 25 + b^2$

$\Rightarrow b^2 = 50$

By using values of a^2 and b^2 in equation (i)

We get $\frac{x^2}{25} - \frac{y^2}{50} = 1$

$\Rightarrow 2x^2 - y^2 = 50$

Example 2. Find the equation of hyperbola with centre at origin and transverse axis is along y-axis, such that latus rectum is 12 units long and eccentricity is 2.

Solution: $\because$ Centre is at origin and transverse axis along y-axis

$\therefore$ its equation will be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

We have $e = 2$ and latus rectum $= 12$

i.e., $\frac{2b^2}{a} = 12$

$\Rightarrow b^2 = 6a \qquad \ldots\text{(ii)}$

We know that

$c^2 = a^2 + b^2$

i.e., $a^2e^2 = a^2 + 6a \qquad (\because c = ae)$

$\Rightarrow 4a^2 = a^2 + 6a$

$\Rightarrow 3a^2 - 6a = 0$

$\Rightarrow 3a(a - 2) = 0$

$\Rightarrow a = 0$ or $a = 2$

Neglecting $a = 0$

We have $a = 2$ or $a^2 = 4$

So, from equation (ii)

we get $b^2 = 12$

By using values of a and b in equation (i)

$$\frac{y^2}{4} - \frac{x^2}{12} = 1$$

or $3y^2 - x^2 = 12$

(d) When focus, eccentricity and centre are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola with centre $(2, 3)$ and transverse axis parallel to x-axis, such that a focus is $(6, 3)$ and eccentricity is $\sqrt{5}$.

Solution: $\because$ Centre is not at origin and transverse axis is parallel to x-axis

$\therefore$ equation of hyperbola will be

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

We have

Centre $= (h, k) = (2, 3)$

and Focus $= (h + c, k) = (6, 3)$

i.e., $(2 + c, 3) = (6, 3)$

$\Rightarrow$ $c = 4$

Now, $c = ae$

i.e., $a = \frac{4}{\sqrt{5}}$ $\qquad (\because e = \sqrt{5})$

We know that

$$c^2 = a^2 + b^2$$

i.e., $16 = \frac{16}{5} + b^2$

$\Rightarrow$ $b^2 = \frac{80-16}{5}$

$\Rightarrow$ $b^2 = \frac{64}{5}$

By using values of h, k, a^2 and b^2 in equation (i)

We get

$$\frac{(x-2)^2}{\frac{16}{5}} - \frac{(y-3)^2}{\frac{64}{5}} = 1$$

$$\Rightarrow \frac{5(x-2)^2}{16} - \frac{5(y-3)^2}{64} = 1$$

(e) When focus, centre and directrix are given

The method is explained with the help of the following example.

Example: Find the equation of hyperbola whose centre is $(3, 4)$ and transverse axis is parallel to y-axis, such that one focus is $(3, 12)$ and one equation of directrix is $y = 7$.

Solution: $\because$ Centre is not at origin and transverse axis parallel to y-axis.

$\therefore$ The equation of hyperbola will be

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

We have

Centre $= (h, k) = (3, 4)$

and Focus $= (h, k + c) = (3, 12)$

i.e., $k + c = 12$

$\Rightarrow$ $4 + c = 12$

i.e., $c = 8$

and one equation of directrix is $y = 7$

Comparing it with $y = \frac{a}{e} + k$

We get

$$\frac{a}{e} + 4 = 7$$

$\Rightarrow$ $a = 3e$ ...(ii)

Also, we know that

$c = ae$

i.e., $ae = 8$

By using $a = 3e$ from equation (ii)

we get

$3e^2 = 8$

$\Rightarrow$ $e^2 = \frac{8}{3}$

$\Rightarrow$ $e = \sqrt{\frac{8}{3}}$

So, equation (ii) becomes $a = 3\sqrt{\frac{8}{3}}$

i.e., $a^2 = 24$

We know that

$$c^2 = a^2 + b^2$$

i.e., $64 = 24 + b^2$

$\Rightarrow$ $b^2 = 40$

By using values of h, k, a^2 and b^2 in equation (i)

we get

$$\frac{(y-4)^2}{24} - \frac{(x-3)^2}{40} = 1$$

9.10.3 Convert a given equation to the standard form of equation of a hyperbola, find its elements and sketch the graph

As we have studied in section 9.10 that the general equation of hyperbola when transverse and conjugate axis are parallel to coordinate axes is

$$Ax^2 + By^2 + Gx + Fy + C = 0$$

Where A and B are non-zero and have opposite sign. Also, A, B, G, F and C are real numbers.

This general equation can be converted into standard forms by the method of completing square which will be explained in the following examples.

Technique for graphing hyperbolas

Graphs of hyperbolas from their standard equations can be drawn by using the following steps.

1. Determine whether the transverse axis is along or parallel to *x*-axis or *y*-axis which can be determined by checking the sign of x^2-term or y^2-term. In case of positive x^2-term, the transverse axis will be along or parallel to *x*-axis.
   In case of positive y^2-term, the transverse axis will be along or parallel to y-axis.
2. Determine the values of *a* and *b* and draw a rectangle extending a units on either side of the centre along the transverse axis and *b* units on either side of the centre along the conjugate axis.
3. Draw the asymptotes along the diagonals of the rectangle.
4. Using the rectangle and the asymptotes as guide draw the graph of hyperbola.

Example 1. Find the eccentricity, foci, vertices and directrices of hyperbola $9x^2 - 16y^2 - 144 = 0$. Also draw its graph.

Solution: First of all we convert the given equation into the standard form.

Given hyperbola: $9x^2 - 16y^2 - 144 = 0$

or $9x^2 - 16y^2 = 144$

Dividing both sides by 144

We get

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Here, centre is origin and the transverse axis is along x-axis with

$a^2 = 16$ and $b^2 = 9$

So, $a = 4$ and $b = 3$

We know that

$c^2 = a^2 + b^2$

i.e., $c^2 = 16 + 9$

$\Rightarrow$ $c^2 = 25$ or $c = 5$

Now, $c = ae$

i.e., $5 = 4e$ $\Rightarrow$ $e = \frac{5}{4}$

$\because$ Major axis is along x-axis.

$\therefore$ coordinates of foci $= (\pm c, 0)$

$= (\pm 5, 0)$

and coordinates of vertices $= (\pm a, 0)$

$= (\pm 4, 0)$

Equation of directrices will be

$$x = \pm\frac{a}{e}$$

or $x = \pm\frac{16}{5}$ i.e., $x = \pm\frac{4}{\frac{5}{4}}$

Graph of Hyperbola

Standard form of given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

By using the steps of drawing graph, we sketch the graph as show in Fig. 9.65.

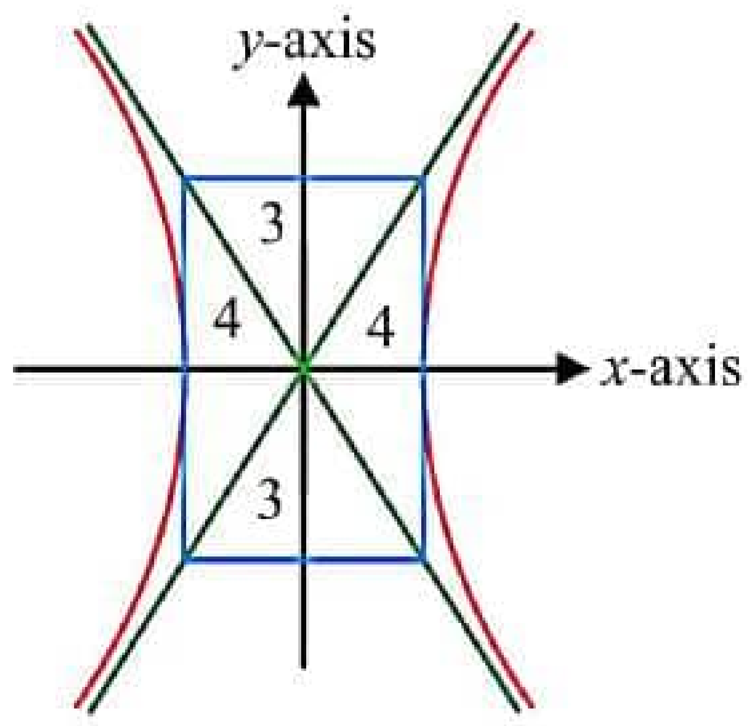


Fig 9.65

Example 2. Find centre, foci, eccentricity and vertices of hyperbola

$16y^2 - 9x^2 + 36x + 64y - 116 = 0$. Also draw its graph.

Solution: We first convert the given hyperbola in standard form.

Given hyperbola: $16y^2 - 9x^2 + 36x + 64y - 116 = 0$

By re-arranging the terms

we get $(16y^2 + 64y) - (9x^2 - 36x) = 116$

or $\quad 16(y^2 + 4y) - 9(x^2 - 4x) = 116$

or $\quad 16(y^2 + 4y + 4) - 9(x^2 - 4x + 4) = 116 + 64 - 36$

or $\quad 16(y + 2)^2 - 9(x - 2)^2 = 144$

Dividing both sides by 144

we get

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$$

Comparing this equation with

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

We get $h = 2, k = -2, a^2 = 9$ and $b^2 = 16$.

and transverse axis is parallel to y-axis.

We know that

$$c^2 = a^2 + b^2$$

i.e., $\quad c^2 = 9 + 16 = 25$

So, $\quad c = 25, a = 3$ and $b = 4$

Now, $\quad c = ae$

i.e., $\quad 5 = 3e \quad \Rightarrow \quad e = \frac{5}{3}$

Now, $\quad$ centre $= (h, k) = (2, -2)$

Foci $= (h, k \pm c) = (2, -2 \pm 5)$

So, $\quad$ Foci are $(2, 3)$ and $(2, -7)$

and $\quad$ vertices $= (h, k \pm a) = (2, -2 \pm 3)$

So, $\quad$ vertices are $(2, 1)$ and $(2, -5)$

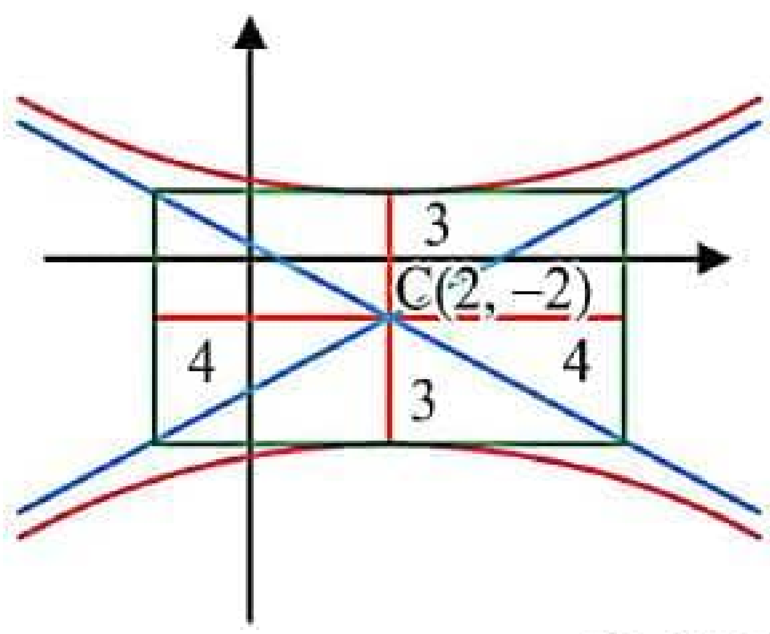


Fig 9.66

Graph of Hyperbola

By using the steps of drawing graph, we draw the graph as shown in the figure 9.66.

Exercise 9.5

1. Find the equation of the hyperbola with centre at the origin satisfying the following conditions.

   (i) Transverse and conjugate axes are 16 and 12 respectively. Also, transverse axis is along y-axis.

(ii) Hyperbola passes through $\left(\frac{3\sqrt{17}}{4}, 1\right)$ and $(3, 0)$ with transverse axis along x-axis.

(iii) Transverse axis of length 8 units and along y-axis where eccentricity is $\sqrt{5}$.

(iv) Transverse axis along x-axis with latus rectum $= 10$ units and eccentricity $= \frac{3}{2}$.

(v) Focus $(5, 0)$, directrix $x = 2$.

(vi) Eccentricity $= 3$ and focus $(8, 0)$.

(vii) Eccentricity $= 2$ and vertex $= (0, 4)$.

2. Find equation of the hyperbola with centre $(1, 3)$ and satisfying the following condition.

(i) Focus is $(2, 3)$ and eccentricity is $\sqrt{3}$, whereas transverse axis is parallel to x-axis.

(ii) Focus is $(4, 5)$ and an equation of directrix is $y = 1$ where transverse axis is parallel to y-axis.

3. Find eccentricity, foci, vertices and latus rectum of each of the following. Also, draw graph.

(i) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (ii) $\frac{y^2}{5} - \frac{x^2}{4} = 1$

(iii) $9x^2 - y^2 + 1 = 0$ (iv) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

4. Find centre, foci, eccentricity, vertices and equations of directrices. Also draw the graph.

(i) $\frac{(x-5)^2}{9} - \frac{(y+3)^2}{16} = 1$ (ii) $\frac{(y-4)^2}{36} - \frac{(x+5)^2}{64} = 1$

(iii) $9x^2 - 4y^2 + 36x + 8y - 4 = 0$

(iv) $25x^2 - 150x - 9y^2 + 72y + 306 = 0$

5. Find equation of rectangular hyperbola with centre at origin whose vertices are $(\pm 4, 0)$ and find equation of its conjugate hyperbola. Also, find equations of asymptotes of the rectangular hyperbola.

6. Find the eccentricity of a hyperbola whose latus rectum is double the transverse axis.

7. Show that the eccentricities e_1 and e_2 of the two conjugate hyperbolas satisfy the relation $e_1^2 + e_2^2 = e_1^2 e_2^2$.

9.11 Equation of Tangent and Normal of a Hyperbola

In this section, we will discuss about the tangent and normal to a hyperbola along with their conditions and equations.

9.11.1 Recognize tangent and normal to a hyperbola

In the figure line l is tangent to the hyperbola as the line touches the hyperbola at a single point, whereas line m is the normal to the hyperbola as it is perpendicular to the tangent at the point of contact P.

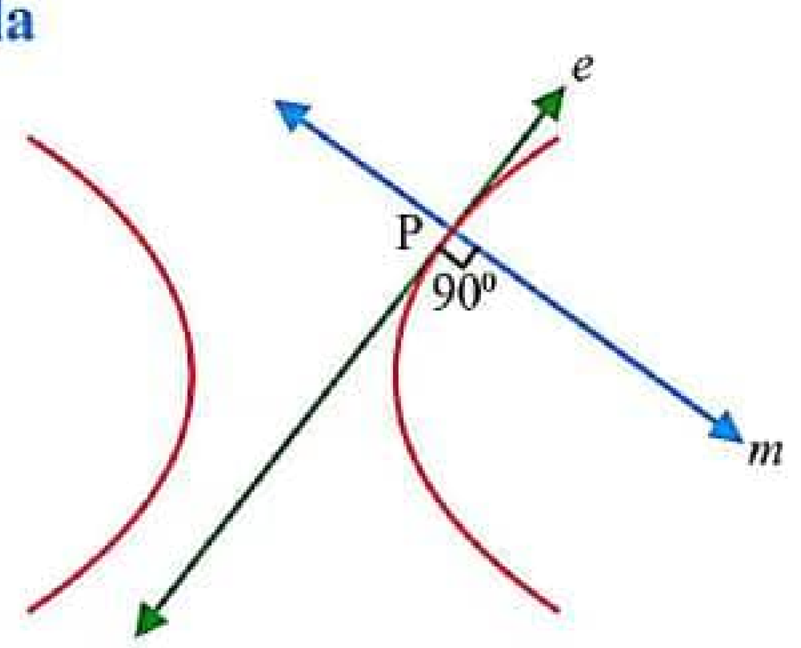


Fig 9.67

9.11.2 Find

- **points of intersection of a hyperbola with a line including the condition of tangency,**
- **the equation of a tangent in slope form.**

Consider a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots\text{(i)}$

and a line $\quad y = mx + c \quad \ldots\text{(ii)}$

Solving both equations simultaneously we get

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$\Rightarrow \quad b^2x^2 - a^2m^2x^2 - 2a^2cmx - a^2c^2 = a^2b^2$

or $\quad (b^2 - a^2m^2)x^2 - 2a^2cmx - a^2c^2 - a^2b^2 = 0$

Here, $\Delta = 4a^4c^2m^2 + 4(a^2c^2 + a^2b^2)(b^2 - a^2m^2)$

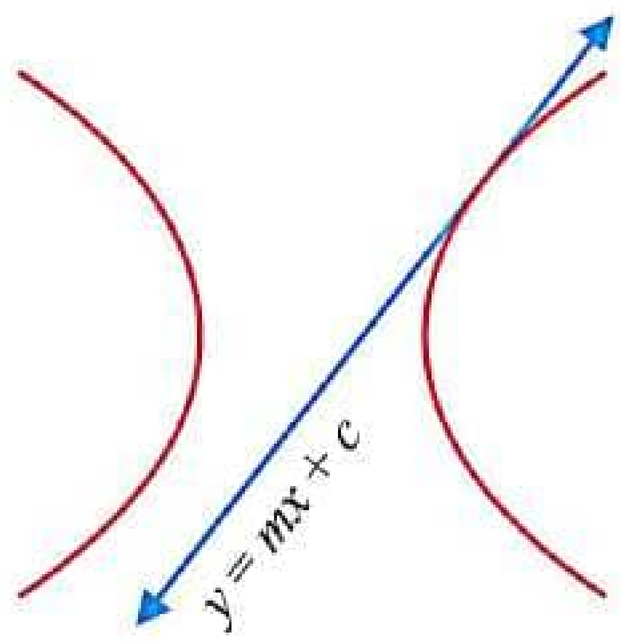


Fig 9.68

By quadratic formula

$$x = \frac{2a^2cm \pm \sqrt{\Delta}}{2a}$$

By using this value of x in equation (ii), we will get value of y, so we will get point of intersection.

The given line will be tangent, if $\Delta = 0$

i.e., $\quad 4a^4c^2m^2 + 4a^2b^2c^2 - 4a^4c^2m^2 + 4a^2b^4 - 4a^4b^2m^2 = 0$

$\Rightarrow \quad 4a^2b^2(c^2 + b^2 - a^2m^2) = 0$

or $\quad c^2 = a^2m^2 - b^2$

$\Rightarrow \quad c = \pm\sqrt{a^2m^2 - b^2}$

This is the condition of tangency when given line is tangent to the hyperbola.

By using this value of m in equation (ii)

we get, $y = mx \pm \sqrt{a^2m^2 - b^2}$

This is the equation of tangent to the given hyperbola in slope form.

Example 1. For what value of k, will the line $y = kx + 1$ be tangent to the hyperbola $3x^2 - 4y^2 = 12$?

Solution: We have

Hyperbola: $3x^2 - 4y^2 = 12$ ...(i)

and line: $y = kx + 1$ ...(ii)

Solving simultaneously,

we get $3x^2 - 4(kx + 1)^2 = 12$

$\Rightarrow$ $3x^2 - 4k^2x^2 - 8kx - 4 - 12 = 0$ ...(iii)

$(3 - 4k^2)x^2 - 8kx - 16 = 0$

$\Delta = 64k^2 - 4(3 - 4k^2)(-16)$

$= 64(k^2 + 3 - 4k^2)$

$= 64(3 - 3k^2)$

The given line will be tangent to the given hyperbola

if $\Delta = 0$

i.e., $64(3 - 3k^2) = 0$

$\Rightarrow$ $k^2 = 1$

$\Rightarrow$ $k = \pm 1$

This is the required value of k.

Example 2. Find the equation of tangent to the hyperbola $2x^2 - 3y^2 = 6$ whose slope is 2.

Solution: We have

Slope $= m = 2$

and hyperbola: $2x^2 - 3y^2 = 6$

i.e., $\frac{x^2}{3} - \frac{y^2}{2} = 1$

Here $a^2 = 3$ and $b^2 = 2$

We know that the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

By using values, we get

$y = 2x \pm \sqrt{12 - 2}$

$y = 2x \pm \sqrt{10}$

This is the required equation of tangent.

9.11.3 Find the equation of a tangent and a normal to a hyperbola at a point

Consider a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(i)

Let $P(x_1, y_1)$ be a point of this hyperbola i.e., $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ...(ii)

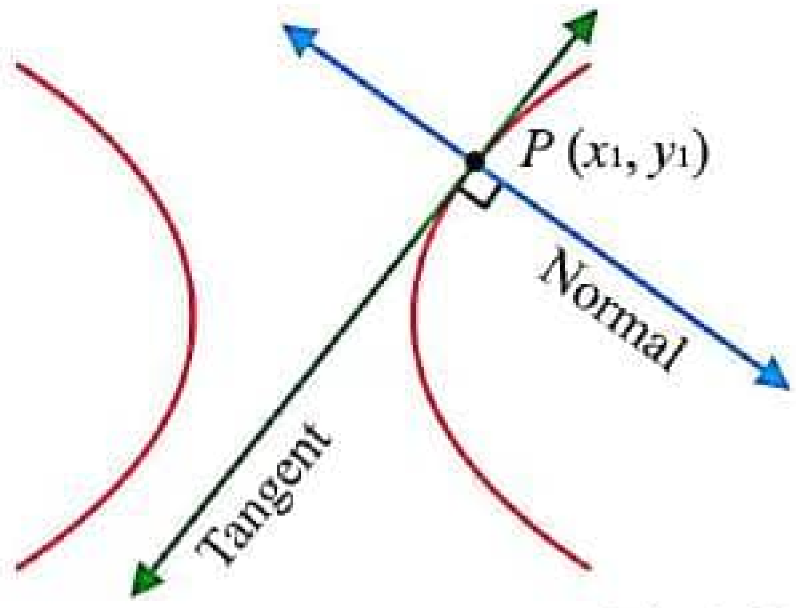


Fig 9.69

Differentiating equation (i) with respect to x, we get

$$\frac{2x}{a^2} - \frac{2y\frac{dy}{dx}}{b^2} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2}{2y} \times \frac{2x}{a^2} = \frac{b^2x}{a^2y}$$

Now slope of tangent to the given hyperbola at (x_1, y_1) is

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{b^2x_1}{a^2y_1}$$

By point slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$\Rightarrow$ $a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$

$\Rightarrow$ $b^2xx_1 - a^2yy_1 - b^2x_1^2 + a^2y_1^2 = 0$

Dividing both sides by a^2b^2

we get,

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = 0$$

i.e., $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ (Using equation (ii)

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

This is the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1)

$\because$ Normal is perpendicular to the tangent at the point of contact

$\therefore$ Slope of normal $= m' = -\frac{a^2y_1}{b^2x_1}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$

$\Rightarrow \quad b^2x_1y - b^2x_1y_1 = -a^2xy_1 + a^2x_1y_1$

Dividing both sides by x_1y_1

we get,

$$\frac{b^2y}{y_1} - b^2 = -\frac{a^2x}{x_1} + a^2$$

$$\Rightarrow \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

This is the equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Example 1. Find the equations of tangent and normal to $\frac{y^2}{4} - \frac{x^2}{5} = 1$ at $(\sqrt{5}, 2\sqrt{2})$.

Solution: We have

Hyperbola: $\frac{y^2}{4} - \frac{x^2}{5} = 1$

Differentiating w.r.t x

$$\frac{2y\frac{dy}{dx}}{4} - \frac{2x}{5} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{5} \times \frac{2}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{5y}$$

Now slope of tangent at $(\sqrt{5}, 2\sqrt{2}) = m = \left(\frac{dy}{dx}\right)_{(\sqrt{5}, 2\sqrt{2})}$

i.e., $\quad m = \frac{4\sqrt{5}}{5(2\sqrt{2})}$

$$= \sqrt{\frac{2}{5}}$$

By point slope form, the equation of tangent will be

$$y - y_1 = m(x - x_1)$$

i.e., $\quad y - 2\sqrt{2} = \sqrt{\frac{2}{5}}(x - \sqrt{5})$

$\Rightarrow \quad \sqrt{5}y - 2\sqrt{10} = \sqrt{2}x - \sqrt{10}$

$\Rightarrow \quad \sqrt{2}x - \sqrt{5}y + \sqrt{10} = 0$

$\because \quad$ Normal is perpendicular to the tangent at the point of contact

$\therefore$ Slope of normal at $(\sqrt{5}, 2\sqrt{2}) = m' = -\frac{1}{m}$

i.e., $m' = -\sqrt{\frac{5}{2}}$

By point-slope form the equation of normal will be

$$y - y_1 = m'(x - x_1)$$

i.e., $y - 2\sqrt{2} = -\sqrt{\frac{5}{2}}(x - \sqrt{5})$

$\Rightarrow$ $\sqrt{2}y - 4 = -\sqrt{5}x + 5$

$\Rightarrow$ $\sqrt{5}x + \sqrt{2}y - 9 = 0$

9.12 Translation and Rotation of Axes

Translation and rotation of axes are the transformations which are commonly used to simplify the equation of a curve and to bring conics in standard forms. We discuss these concepts in detail as under.

9.12.1 Define translation and rotation of axes and demonstrate through examples

The concept of translation is of a transformation in which the location of the geometrical shape is changed but its size, shape or orientation remains same.

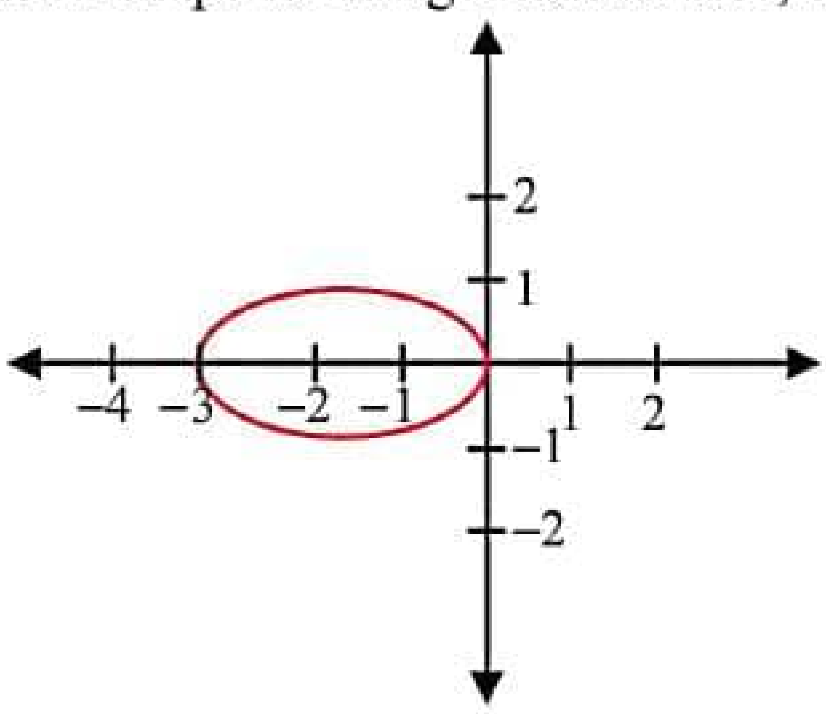


Fig. 9.70

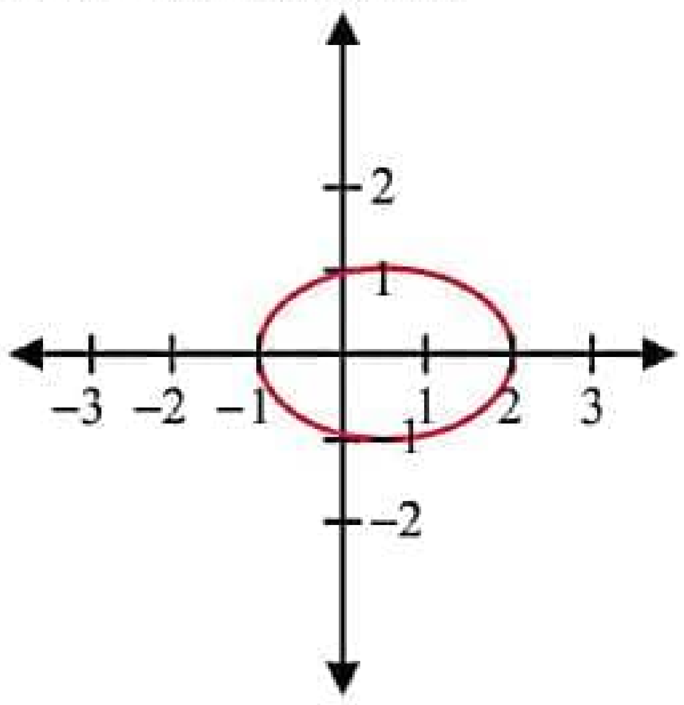


Fig. 9.71

For example, an ellipse in Fig. 9.70 has been translated 2 units to the right as shown in Fig. 9.71.

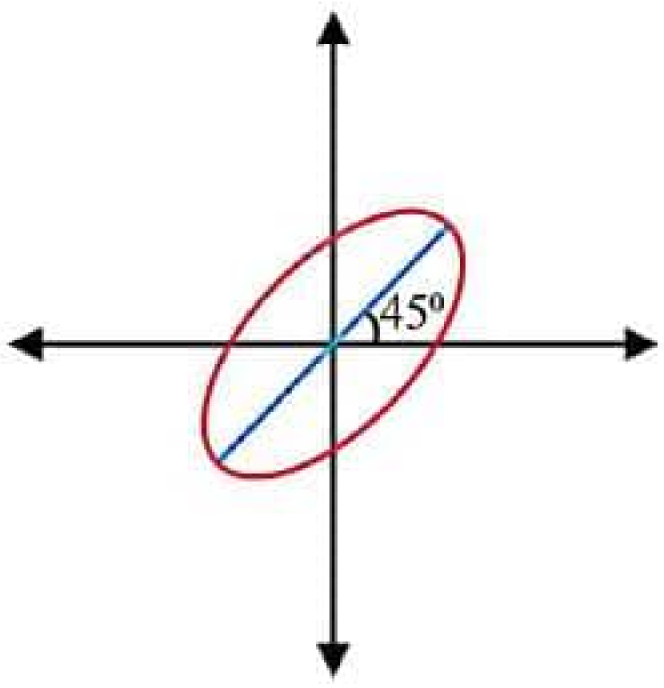


Fig. 9.72

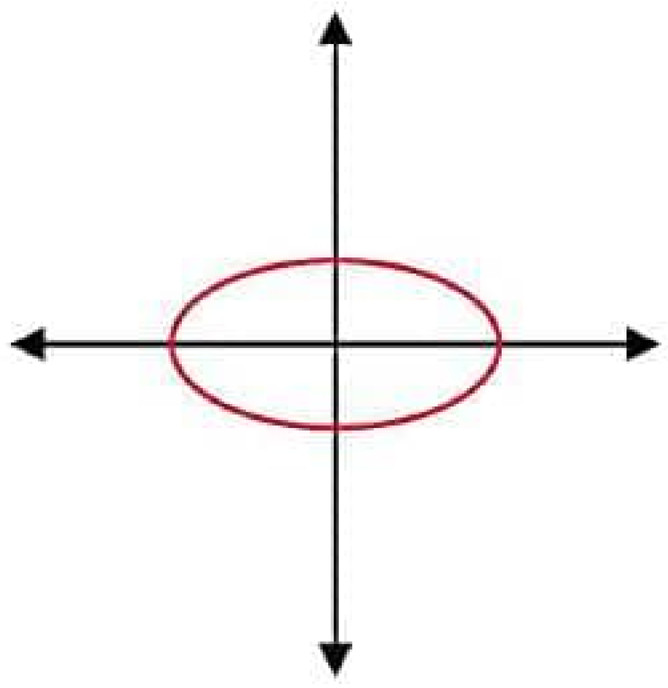


Fig. 9.73

The concept of rotation is also of a transformation in which the location of the geometrical shape is rotated around a fixed point but its size and shape are not changed, for example an ellipse in Fig.9.72 has been rotated 45° clockwise as shown in Fig. 9.73.

Definition: A translation of axes is a transformation between two rectangular coordinate systems in which the origins O and O′ are at different locations but the corresponding axes are parallel and have the same directions as shown in the figure 9.74.

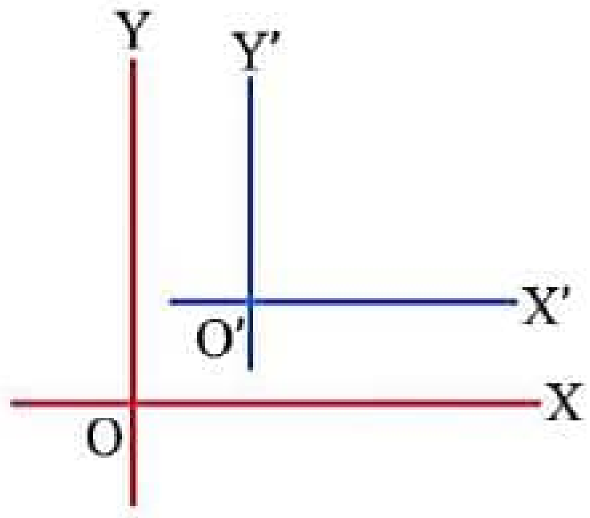


Fig 9.74

Definition: A rotation of axes in the plane is a transformation in which the axes OX and OY of one rectangular system are rotated about the origin O through an angle θ to locate the corresponding axes OX′ and OY′ of other coordinate system as shown in the figure 9.75.

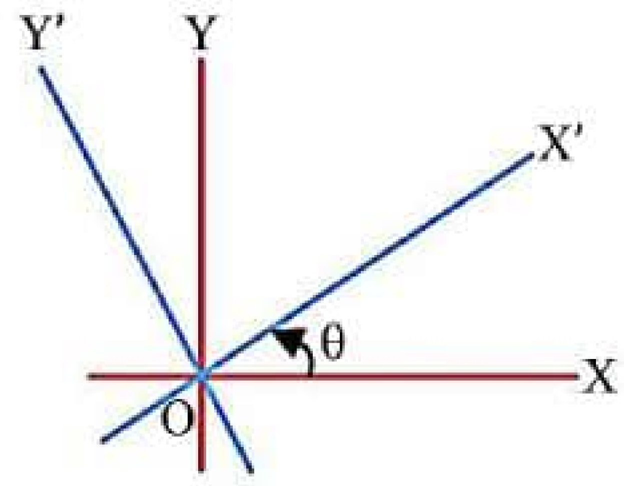


Fig 9.75

9.12.2 Find the equations of transformation for

- translation of axes,
- rotation of axes.

- **Equations of transformation for translation of axes**

In order to obtain the equations of transformation for translation of axes we have translated the axes of an xy −coordinate system to get a new $x'y'$ − coordinate system whose origin O′ is at the point (h, k) as shown in Fig. 9.76.

As a result, a point P in the plane will have both (x, y)- coordinates and (x', y')-coordinates as shown in the Fig. 9.77. These coordinates are related by

$$x = x' + h\ , \qquad y = y' + k$$

or $$x' = x - h\ , \qquad y' = y - k$$

These equations are called the equations of transformation for the translation of axes.

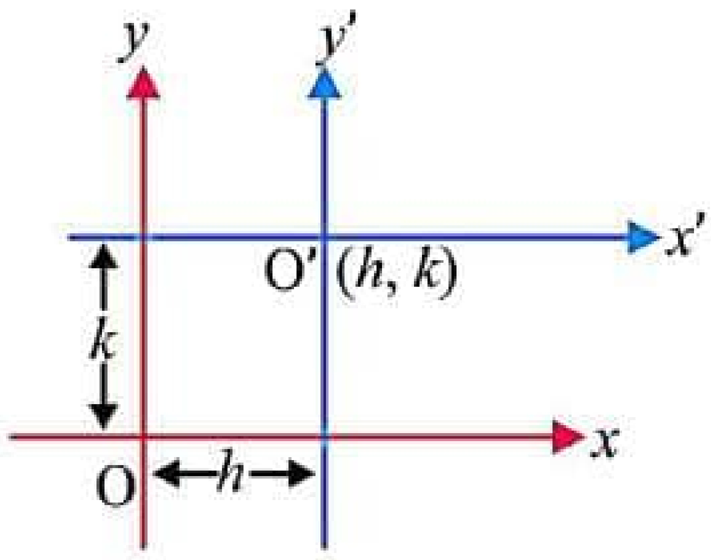


Fig. 9.76

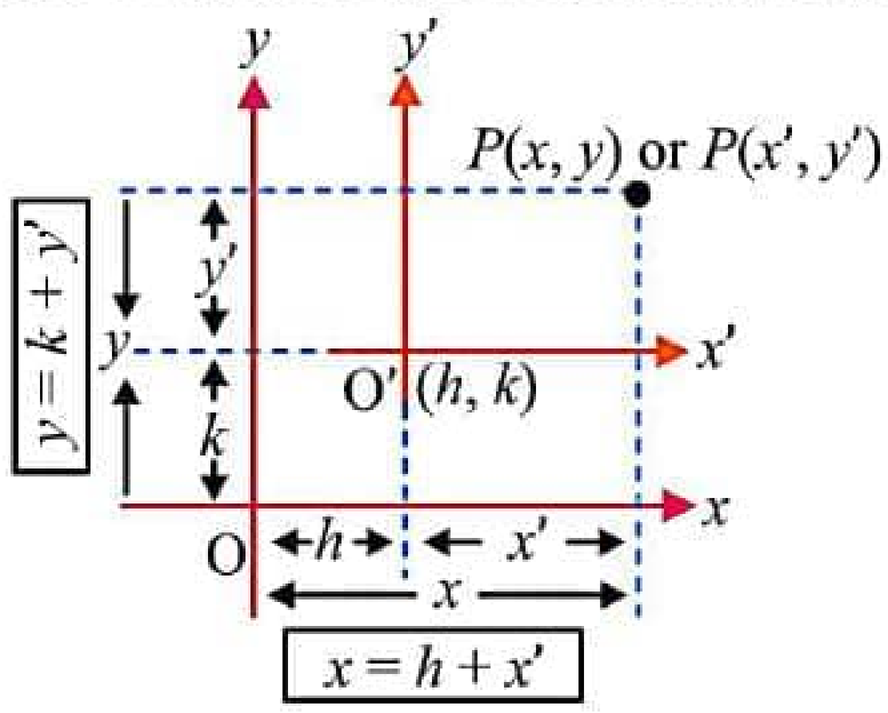


Fig. 9.77

- **Equations of transformation for rotation of axes**

In order to get the equations of transformation for rotation of axes, the axes of an $xy-$ coordinate system have been rotated about the origin through an angle θ to produce a new $x'y'-$ coordinate system as shown in the Fig. 9.78.

As a result, any point P in the plane will have both $P(x,y)$- coordinates and $P(x',y')$- coordinates as shown in the Fig. 9.79.

In order to relate these coordinates, we suppose r as the distance from the common origin to the point P and let α be the angle of $\overline{OP}$ from x'-axis as shown in the Fig.9.79.

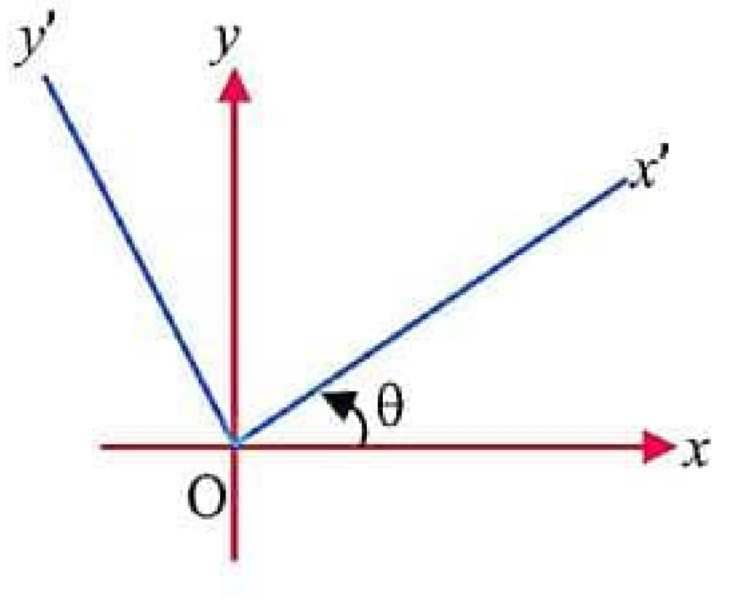


Fig. 9.78

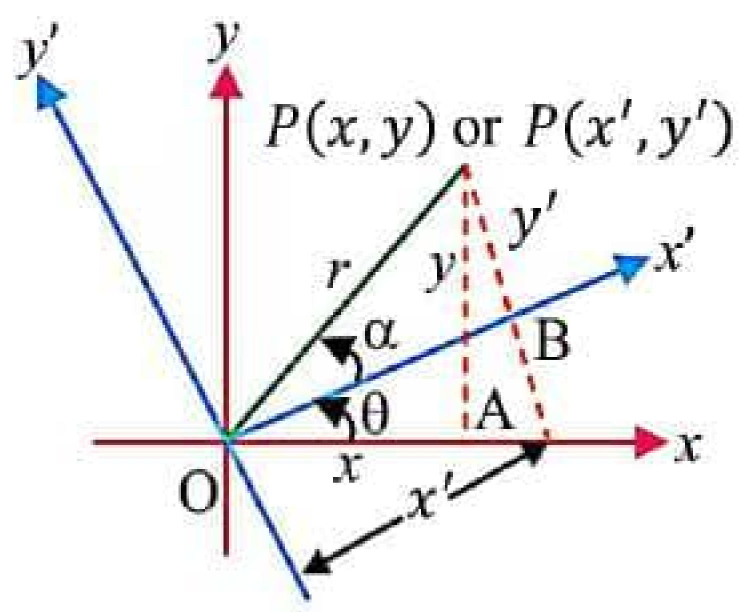


Fig. 9.79

From figure 9.79, in ΔOAP

$$\cos(\theta+\alpha)=\frac{x}{r} \text{ and } \sin(\theta+\alpha)=\frac{y}{r}$$

or $\quad x=r\cos(\theta+\alpha)$ ...(i)

or $\quad y=r\sin(\theta+\alpha)$ ...(ii)

In ΔOBP

$\quad x'=r\cos\alpha$ ...(iii)

and $\quad y'=r\sin\alpha$ ...(iv)

Using trigonometric identities equation (i) and equation (ii) become

$$x=r\cos\theta\cos\alpha-r\sin\theta\sin\alpha$$

and $\quad y=r\sin\theta\cos\alpha+r\cos\theta\sin\alpha$

By using equation (iii) and (iv)

we get

$$\left.\begin{aligned}x&=x'\cos\theta-y'\sin\theta\\ y&=x'\sin\theta+y'\cos\theta\end{aligned}\right\}$$

These equations are called the equations of transformation for rotation of axes.

9.12.3 Find the transformed equation by using translation or rotation of axes

The method of finding the transformed equation by using translation or rotation of axes is explained with the help of the following examples.

Example 1. Find the transformed equation of parabola $(y+5)^2 = 4(x-3)$ when axes are translated with new origin $(3,-5)$.

Solution: Given parabola is $(y+5)^2 = 4(x-3)$ ...(i)

Shifting the origin to $(3,-5)$ and keeping the axes in parallel position.

Let (X, Y) be the new coordinates of any point $P(x, y)$ after shifting the origin.

By equations of transformation

$$x' = x - h \text{ and } y' = y - k$$

Here $(x', y') = (X, Y)$

and $(h, k) = (3, -5)$

So, we get

$$X = x - 3 \text{ and } Y = y + 5$$

So, equation (i) becomes

$$Y^2 = 4$$

This is the required transformed equation.

Example 2. Find the transformed equation of $5x^2 - 6xy + 5y^2 - 8 = 0$ when the axes are rotated through an angle of $45°$.

Solution: Given equation is $5x^2 - 6xy + 5y^2 - 8 = 0$ ...(i)

Now, we rotate the axes about the origin through an angle of $\theta = 45°$

Let (X, Y) be the new coordinates of any point $P(x, y)$ after rotation

By equations of transformation, we have

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$

Here $(x', y') = (X, Y)$

and $\theta = 45°$

So, we get

$$x = X\cos 45° \, Y\sin 45° = \frac{X - Y}{\sqrt{2}}$$

and

$$y = X\sin 45° + Y\cos 45° = \frac{X + Y}{\sqrt{2}}$$

Substituting these values in equation (i), we get

$$5\left(\frac{X-Y}{\sqrt{2}}\right)^2 - 6\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 5\left(\frac{X+Y}{\sqrt{2}}\right)^2 - 8 = 0$$

$$\Rightarrow \frac{5}{2}(X^2 - 2XY + Y^2) - 3(X^2 - Y^2) + \frac{5}{2}(X^2 + 2XY + Y^2) - 8 = 0$$

$$\Rightarrow \frac{5}{2}(2X^2 + 2Y^2) - 3X^2 + 3Y^2 - 8 = 0$$

$$\Rightarrow \quad 2X^2 + 8Y^2 - 8 = 0$$

or $\quad X^2 + 4y^2 = 4$

This is the required transformed equation.

9.12.4 Find new origin and new axes referred to old origin and old axes

Let O be the origin of *xy*-coordinate system as shown in the figure 9.80.

A new XY-coordinate system is introduced with new origin $O'(h, k)$. This system is translated h units in the *x*-direction and k units in the y-direction and then rotated anticlockwise by θ radians as shown in the figure.

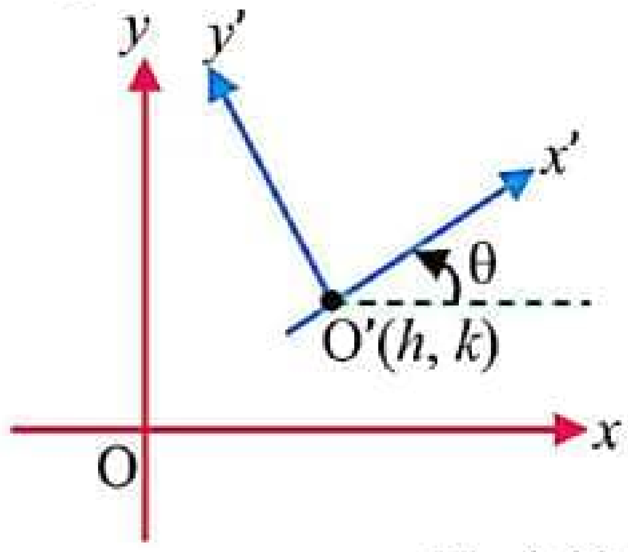


Fig 9.80

The relations among x, y, X, Y and θ are given below

$$x = (X + h)\cos\theta - (Y + k)\sin\theta$$

and $\quad y = (X + h)\sin\theta + (Y + k)\cos\theta$

Example 1. Find new origin in O' and new axes (X-axis and Y-axis) with respect to xy-coordinate system if it is translated 5 units to the right, 3 units down and rotated $\frac{\pi}{4}$ radius anticlockwise.

Solution: Here $\quad h = 5$ and $k = -3$

So, new origin $= (h, k) = (5, -3)$

Here inclination of X-axis $= \theta = \frac{\pi}{4}$

So, slope of X-axis $= \tan 45°$

$= 1$

By point slope form equation of X-axis will be

$$y - (-3) = 1(x - 5)$$

$$\Rightarrow \quad y + 3 = x - 5$$

$$\Rightarrow \quad x - y - 8 = 0$$

Now, inclination of Y-axis $= \theta = \frac{\pi}{4} + \frac{\pi}{2}$

or $\quad \theta = \frac{3\pi}{4}$

Its slope $= \tan\theta$

$= \tan\frac{3\pi}{4} = -1$

By point-slope form, the equation of Y-axis will be

$$y - (-3) = -1(x - 5)$$

$\Rightarrow \quad y + 3 = -x + 5$

$\Rightarrow \quad x + y - 2 = 0$

So new origin = $(5, -3)$,

Equation of X-axis is: $x - y - 8 = 0$

and equation of Y-axis is: $x + y - 2 = 0$

Example 2. Find new coordinates of $P(4, 5)$ if new origin is $(2, 3)$ and XY-coordinate system is rotated with $\frac{\pi}{4}$ radians anticlockwise from *xy*-coordinate system.

Solution: Here $\quad (x, y) = (4, 5)$

$(h, k) = (2, 3)$

and $\quad \theta = \frac{\pi}{4}$

By the equation of transformation

$x = (X + h)\cos\theta - (Y + k)\sin\theta$ and $\qquad y = (X + h)\sin\theta + (Y + k)\cos\theta$

i.e., $\quad 4 = \frac{(X+2)}{\sqrt{2}} - \frac{(Y+3)}{\sqrt{2}} \qquad$ i.e., $\quad 5 = \frac{(X+2)}{\sqrt{2}} + \frac{(Y+3)}{\sqrt{2}}$

$\Rightarrow \quad 4\sqrt{2} = X - Y - 1 \qquad \ldots$(i) $\qquad \Rightarrow \quad 5\sqrt{2} = X + Y + 5 \qquad \ldots$(ii)

Adding equation (i) and (ii), we get

$9\sqrt{2} = 2X + 4$

$\Rightarrow \quad X = \frac{9\sqrt{2}-4}{2}$

By using this value of X in equation (ii), we get

$5\sqrt{2} = \frac{9\sqrt{2} - 4}{2} + Y + 5$

$\Rightarrow \quad 5\sqrt{2} - 5 - \frac{(9\sqrt{2}-4)}{2} = y$

$\Rightarrow \quad y = \frac{\sqrt{2}-6}{2}$

So, new coordinates of $P(4, 5)$ are $\left(\frac{9\sqrt{2}-4}{2}, \frac{\sqrt{2}-6}{2}\right)$

9.12.5 Find the angle through which the axes be rotated about the origin so that the product term *xy* is removed from the transformed equations

If we remove *xy*-term from the second degree equation in x and y then the equation is reduced to familiar form of equation of conic.

The following theorem tells how to determine an appropriate rotation of axes to eliminate the *xy*-term of a second degree equation in x and y.

Theorem: If the equation $Ax^2 + By^2 + Hxy + Gx + Fy + C = 0$ is such that $H \neq 0$ and if an XY-coordinate system is obtained by rotating the xy-axes through an angle θ satisfying.

$$\cot 2\theta = \frac{A - B}{H}$$

then in XY-coordinates, the given equation will have the form

$$A'x^2 + B'y^2 + G'x + F'y + C' = 0$$

Example: Identify and sketch the curve $xy = 1$.

Solution: We have $xy = 1$ ...(i)

Comparing given equation with $Ax^2 + By^2 + Hxy + Gx + Fy + C = 0$, we get

$A = 0, B = 0$ and $H = 1$

Now, $\cot 2\theta = \frac{A-B}{H} = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4} = 45°$

By the equation of transformations

$x = X\cos\theta - Y\sin\theta$ and

i.e., $\quad x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$

$y = X\sin\theta + Y\cos\theta$

$\Rightarrow \quad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$

By substituting these values in equation (i)

We get,

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1$$

$$\Rightarrow \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1$$

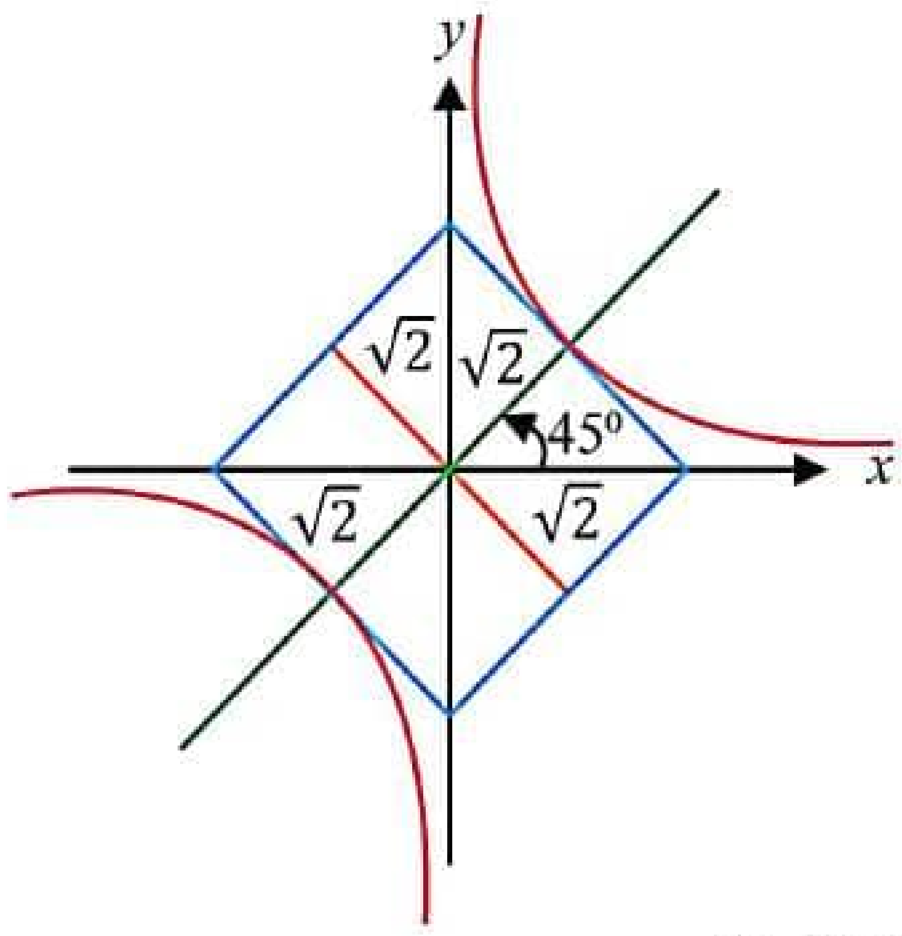


Fig 9.81

This is the equation of rectangular hyperbola with centre at origin and rotation of 45°.

Here $\quad a^2 = 2$ and $b^2 = 2$

So, $\quad c^2 = 4 \Rightarrow c = 2$

Here vertices are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ in XY-coordinate system.

The graph is sketched as shown in Fig. 9.81.

Exercise 9.6

1. For what value of k, the line $y = 2kx$ will be tangent to $2x^2 - 5y^2 = 10$.

2. Find the condition when the line $y = mx + c$ is tangent to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

3. Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 12$ when slope is 3.
4. Find the equation of tangent and normal to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ at (x_1, y_1).
5. Find the equation of tangent and normal to $\frac{x^2}{5} - \frac{y^2}{7} = 1$ at $(2\sqrt{5}, \sqrt{7})$.
6. Find the transformed equation of $\frac{(x-6)^2}{25} + \frac{(y+7)^2}{16} = 1$ when axes are translated with new origin $(6, -7)$.
7. If xy-axes are rotated through given angle θ then find the new coordinates of given point P

   (i) $(2, 3), \theta = 60°$ (ii) $(6, 7), \theta = 45°$ (iii) $(-4, 6), \theta = 30°$
8. Find new origin O' and new XY-axes with respect to xy-coordinate system if it is translated 6 units to the left, 5 units up and rotated $\frac{\pi}{6}$ radians anticlockwise.
9. Find new coordinates of $P(4, 5)$ if new origin is $(1, 2)$ and XY-coordinate system is rotated with $\frac{\pi}{6}$ radians anticlockwise from xy-coordinate system.
10. Identity and sketch the curve $xy = 9$.
11. Through which angle the axes be rotated about origin so that the transformed equation of $9x^2 + 12xy + 4y^2 - x - y = 0$ does not contain the term involving XY.

Review Exercise 9

1. Tick the correct option.

(i) If the eccentricity is zero, then the conic is ----------------

(a) parabola (b) ellipse (c) circle (d) hyperbola

(ii) The focus of parabola $x^2 = -16y$ is ------------

(a) $(0, 0)$ (b) $(4, 0)$ (c) $(-4, 0)$ (d) $(0, -4)$

(iii) The latus rectum and vertex of $(y - 3)^2 = -8(x + 4)$ is ------------

(a) $-8, (3, -4)$ (b) $8, (3, -4)$ (c) $4, (-3, -4)$ (d) $8, (-4, 3)$

(iv) The equation of tangent at $(4, 6)$ to the parabola $y^2 = 9x$ is --------------

(a) $6y = \frac{9}{2}(x + 4)$ (b) $6y = 9(x - 4)$

(c) $4y = \frac{9}{2}(x + 6)$ (d) $3x - 4y + 12 = 0$

(v) The latus rectum of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is -----------

(a) $\frac{5}{32}$ (b) $\frac{32}{5}$ (c) $\frac{50}{4}$ (d) None

(vi) The eccentricity of the conic $\frac{x^2}{5}+\frac{y^2}{4}=1$ is -----------

(a) $\sqrt{5}$ (b) $\frac{1}{\sqrt{5}}$ (c) 5 (d) None

(vii) The centre of ellipse $\frac{(x+5)^2}{10}+\frac{(y-3)^2}{20}=1$ is -----------

(a) $(\sqrt{10},\sqrt{20})$ (b) $(5,3)$ (c) $(-5,3)$ (d) None

(viii) The equation of directrix for the conic $\frac{x^2}{4}+\frac{y^2}{2}=1$ is -----------

(a) $x=\pm\frac{4}{\sqrt{2}}$ (b) $x=\pm\frac{\sqrt{5}}{4}$ (c) $x=\pm\frac{2}{\sqrt{5}}$ (d) $x=\pm 2\sqrt{2}$

(ix) $ax^2+by^2+gx+fy+c=0$ where a,b,g,f and c are real numbers that represents hyperbola if

(a) a and b are non-zero and of same sign

(b) a and b are non-zero and of different sign

(c) either $a=0$ or $b=0$ (d) $a=b=0$

(x) Auxiliary circle of ellipse $\frac{x^2}{6}+\frac{y^2}{5}=1$ is -----------

(a) $x^2+y^2=36$ (b) $x^2+y^2=25$

(c) $x^2+y^2=5$ (d) $x^2+y^2=6$

(xi) The equations of directrices for $\frac{(x-h)^2}{p^2}+\frac{(y-k)^2}{q^2}=1$ are ----------- where $q>p$

(a) $x=\pm\frac{p}{e}$ (b) $x-h=\pm\frac{q}{e}$

(c) $y-k=\pm\frac{q}{e}$ (d) $x-h=\pm\frac{p}{e}$

(xii) The vertices of hyperbola $\frac{x^2}{9}+\frac{y^2}{16}=1$ are -----------

(a) $(\pm5,0)$ (b) $(0,\pm5)$ (c) $(0,\pm4)$ (d) $(\pm4,0)$

(xiii) Conjugate hyperbola to $\frac{x^2}{5}-\frac{y^2}{6}=1$ is -----------

(a) $\frac{x^2}{5}-\frac{y^2}{6}=1$ (b) $\frac{y^2}{6}-\frac{x^2}{5}=1$ (c) $\frac{x^2}{6}-\frac{y^2}{5}=1$ (d) None

(xiv) The eccentricity of rectangular hyperbola is ------------

(a) 1 (b) 2 (c) $\sqrt{3}$ (d) $\sqrt{2}$

(xv) The equation of tangent to $\frac{x^2}{6}+\frac{y^2}{5}=1$ at $(\sqrt{6},0)$ is ------------

(a) $x=6$ (b) $x=\sqrt{6}$ (c) $y=\sqrt{6}$ (d) None

(xvi) For what value of k, $y=k$ is tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ is -----------

(a) ± 3 (b) ± 5 (c) $\pm\frac{7}{5}$ (d) None

(xvii) The equation of tangent to $\frac{x^2}{16}-\frac{y^2}{9}=1$ with slope 2 is ----------------

(a) $y=2x\pm\sqrt{23}$ (b) $y=2x\pm\sqrt{41}$

(c) $x=2y\pm\sqrt{23}$ (d) $y=2x\pm\sqrt{55}$

(xviii) The equation of $xy=c^2$ represents

(a) parabola (b) ellipse (c) hyperbola (d) circle

(xix) If origin is shifted to $(2,3)$ then coordinates of $(5,6)$ are -----------

(a) $(2,2)$ (b) $(3,3)$ (c) $(4,4)$ (d) None

(xx) If xy-coordinate system is rotated at angle of $\frac{\pi}{4}$ transformation for abscissa is

(a) $x=\frac{x'}{2}-\frac{y'}{\sqrt{2}}$ (b) $x=\frac{x'}{\sqrt{2}}-\frac{y'}{\sqrt{2}}$

(c) $x=\frac{x'}{\sqrt{3}}-\frac{y'}{\sqrt{3}}$ (d) None

2. Find the foci, vertices and directrices for the conic

(a) $\frac{(x-5)^2}{25}+\frac{(y+3)^2}{16}=1$ (b) $\frac{(x+4)^2}{9}-\frac{(y+7)^2}{16}=1$

3. Find the condition of tangency the line $y=x+c$ is tangent to the conic

(i) $y^2=10x$ (ii) $2x^2+3y^2=6$

(iii) $5x^2-7y^2=35$

4. Find transformed equation of $\frac{(x+5)^2}{7}-\frac{(y-3)^2}{5}=1$ when new origin is $(-5,3)$.

5. If xy-axes are rotated through angle θ, find coordinates of P if new coordinates is $(-2,7), \theta=45°$.

Unit 10 Differential Equations

10.1 Introduction

In previous chapters of differentiation, we discussed how to differentiate a given function f with respect to an independent variable i.e., how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in the chapter of integration, we discussed how to find a function f whose derivative is the function g, which may also be formulated as follows.

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x) \text{ where } y = f(x) \quad \text{... (i)}$$

An equation of the form (i) is known as differential equation. It is defined as an equation containing the derivatives of one or more dependent variables with respect to one independent variables.

10.1.1 Define ordinary differential equation (DE), order of a DE, degree of a DE, solution of a DE – general solution and particular solution

An equation involving derivatives ordinary derivatives of one or more dependent variables with respect to a single independent variable is called ordinary differential equation.

(i) $\frac{dy}{dx} + 5y = e^x$ (ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$

(iii) $\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$ (iv) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + xy = 0$

Order and degree of the differential equation

The order of a differential equation is the order of the highest derivative appearing in it.

The degree of the differential equation is the degree of the highest order derivative occurring in it, after the equation has been expressed in a form free from radicals and non-integer powers of derivatives.

Solution of differential equation

A solution of a differential equation is a relation between the variables free from derivatives, such that this relation and the derivatives obtained from it satisfies the given differential equation.

Example: Find the order and degree of following differential equation.

(i) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 + x = e^x$

The order of differential equation is 2 and its degree is 1.

(ii) $\left(\frac{d^3y}{dx^3}\right)^2 + 3\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} + y = x$

The order of differential equation is 3 and its degree is 2.

(iii) $\left(\frac{dy}{dx}\right)^3 = \sqrt{1 + \frac{d^2y}{dx^2}}$

The equation contains fraction power of derivative. First, we reduce it into integer power by squaring the whole equation.

We get

$$\left(\frac{dy}{dx}\right)^6 = 1 + \frac{d^2y}{dx^2}$$

Now the order of differential equation is 2 and its degree is 1.

(iv) $\frac{d^2y}{dx^2} = 2y^{\frac{1}{3}}$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^3 = 2y$$

order of differential equation is 2 and its degree is 3.

Example 1. Show that $y = Ae^{2x}$ is the solution of differential equation $\frac{dy}{dx} - 2y = 0$.

Solution: The given differential equation is

$$\frac{dy}{dx} - 2y = 0 \qquad \text{...(i)}$$

Now, to verify $y = Ae^{2x}$ is the solution of the differential equation. We differentiate y w.r.t x

$$\frac{dy}{dx} = 2Ae^{2x}$$

Now by substituting the values of y and $\frac{dy}{dx}$ in equation (i), we get

$$2Ae^{2x} - 2Ae^{2x} = 0$$
$$0 = 0$$

Hence $y = Ae^{2x}$ is the solution of differential equation $\frac{dy}{dx} - 2y = 0$.

Example 2. Show that $y = A\sin x + B\cos x$ is the solutions of differential equation $\frac{d^2y}{dx^2} + y = 0$.

Verification: The given differential equation is

$$\frac{d^2y}{dx^2} + y = 0 \qquad ...(i)$$

Now to verify, $y = A\sin x + B\cos x$ is the solution of the given differential equation,

We differentiate y w.r.t x

We have

$$\frac{dy}{dx} = \frac{d}{dx}(A\sin x + B\cos x)$$

$$\frac{dy}{dx} = A\cos x - B\sin x$$

Again, differentiate w.r.t x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(A\cos x - B\sin x)$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

Now by substituting the values of y and $\frac{d^2y}{dx^2}$ in equation (i), we get

$$-A\sin x - B\cos x + (A\sin x + B\cos x) = 0$$

$$-A\sin x - B\cos x + A\sin x + B\cos x = 0$$

$$0 = 0$$

Hence, $y = A\sin x + B\cos x$ is the solution of differential equation $\frac{d^2y}{dx^2} + y = 0$.

General and Particular Solution:

The general solution (complete solution) of a differential equation is the one in which the number of arbitrary constants is equal to the order of the differential equation.

A solution obtained from the general solution by giving particular values to the arbitrary constants is called particular solution.

For example, the differential equation $\frac{d^2y}{dx^2} + y = 0$ has the general solutions $y = A\sin x + B\cos x$ whose A & B are arbitrary constants. When we assign fixed values to arbitrary constants according to given condition. For example, at $y(0) = 1$ and $y'(0) = 2$, we get $A = 2$ and $B = 1$, then the solution will be $y = 2\sin x + \cos x$ known as particular solution.

Example 1. Verify that $y = Ae^x + Be^{2x}$ is the solution of differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$. Also, find the particular solution of the equation satisfying the conditions $y(0) = 1$ and $y'(0) = -1$.

Solution: Since $y = Ae^x + Be^{2x}$

$$\frac{dy}{dx} = Ae^x + 2Be^{2x}$$

$$\frac{d^2y}{dx^2} = Ae^x + 4Be^{2x}$$

Putting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

We get $Ae^x + 4Be^{2x} - 3Ae^x - 6Be^{2x} + 2Ae^x + 2Be^{2x} = 0$ ...(i)

$0 = 0$ Hence proved.

To find the particular solution we use the given conditions:

$y(0) = 1$ (mean when $x = 0, y = 1$)

and $y'(0) = -1$ (means when $x = 0, \frac{dy}{dx} = -1$)

$y(0) = 1 \quad \Rightarrow \quad 1 = Ae^0 + Be^0$

$\Rightarrow \quad A + B = 1$ ...(i)

$y'(0) = -1 \quad \Rightarrow \quad -2 = Ae^0 + 2Be^0$

$\Rightarrow \quad A + 2B = -1$ ...(ii)

Solving (i) and (ii) we get $A = 3, B = -2$.

Hence the particular solution of given differential equation is $y = 3e^x - 2e^{2x}$.

10.2 Formation of differential Equation

10.2.1 Demonstrate the concept of formation of a differential equation

If the relation between the dependent variable and independent variable involves some arbitrary constants, we can form a differential equation by eliminating arbitrary constants from the relation by differentiating with respect to the independent variable successively as many times as the number of arbitrary constants. We illustrate by the following examples.

Example: Form the differential equation

(a) $y = A\sin 2x + B\cos 2x$ (b) $y = C_1e^x + C_2e^{-x}$

(c) $y = x + Ce^x$ (d) $x^2 + y^2 = r^2$

Solution: (a) $y = A\sin 2x + B\cos 2x$ ...(i)

As there are two arbitrary constants, so we differentiate two times. Differentiate (i) with respect to x

$$\frac{dy}{dx} = A\cos 2x\,(2) - B\sin 2x\,(2) \quad \text{...(ii)}$$

Again, differentiate with respect to x

$$\frac{d^2y}{dx^2} = -A\sin 2x\,(4) - B\cos 2x\,(4)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -4(A\sin 2x + B\cos 2x)$$

$$\frac{d^2y}{dx^2} = -4y \qquad \because y = A\sin 2x + B\cos 2x$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution: (b) $y = C_1e^x + C_2e^{-x}$ ...(i)

As there are two arbitrary constants, so we differentiate two times. Differentiate (i) with w.r.t x

$$\frac{dy}{dx} = C_1e^x - C_2e^{-x} \qquad \text{...(ii)}$$

Again, differentiate w.r.t x

$$\frac{d^2y}{dx^2} = C_1e^x - C_2e^{-x}(-1) = c_1e^x + c_2e^{-x} \qquad \text{...(iii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \qquad \text{(Using i)}$$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} - y = 0$ is required differential equation.

Solution: (c) $y = x + Ce^x$ ...(i)

As there are only one constant, so we differentiate one time. Differentiate (i) with w.r.t x

$$\frac{dy}{dx} = 1 + Ce^x \qquad \text{...(ii)}$$

Now by using equation (i)

$$\frac{dy}{dx} = 1 + y - x$$

This is the required differential equation.

Solution: (d) $x^2 + y^2 = r^2$ ...(i)

Since there is only one arbitrary constant, so we differentiate one time. Differentiate (i) w.r.t x

$$\frac{d}{dx}(x)^2 + \frac{d}{dx}(y)^2 = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

which is the required differential equation.

Exercise 10.1

1. Find the order and degree of each of the following differential equation.

   (i) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + xy = 0$ (ii) $x^3dx + y^3dy = 0$

   (iii) $\frac{dy}{dx} = \sqrt[3]{\left(\frac{d^2y}{dx^2} + 1\right)^2}$ (iv) $\frac{d^3y}{dx^3} - 5\left(\frac{d^2y}{dx^2}\right)^3 + 7\left(\frac{dy}{dx}\right)^8 = 0$

   (v) $\frac{d^4y}{dx^4} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$

2. Show that $y = x - x\ln x$ is the solution of the differential equation $x\frac{dy}{dx} + x - y = 0$.

3. Show that $y = Ae^{2x} + Be^{3x}$ is the general solution of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

4. Obtain the differential equation by eliminating arbitrary constant from the relation.

   (i) $y = A\cos x + B\sin x$ (ii) $y = A\sin(x + 1)$

   (iii) $y = ax^2 + bx$ (iv) $y = C_1e^x + C_2e^{-2x}$

5. Find the particular solution of:

   (i) $\frac{dy}{dx} = \frac{\sqrt{1+\cos y}}{\sin y}, y(3) = \frac{\pi}{2}$, given that $x + 2\sqrt{1+\cos y} + c = 0$ is the general solution of the differential equation.

   (ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0, y(0) = 1, y'(0) = 3$, given that $y = Ae^{2x} + Be^{-x}$ is the general solution of the differential equation.

10.3 Solution of Differential Equation

10.3.1 Solve differential equations of first order and first degree of the form:

- **separable variables,**
- **homogeneous equations,**
- **equations reducible to homogeneous form.**

(i) Separable variables

If the differential equation $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ ...(i)

Where $f(x)$ is the function of x only and $g(y)$ is the function of y only, then we can write equation (i) as below

$$g(y)dy = f(x)dx$$

The equation is in separable variable form. To find the solution we integrate both sides,

i.e., $\int g(y)dy = \int f(x)dx + C$

Where C is an arbitrary constant, called constant of integration.

Example 1. $\frac{dy}{dx} - x^2 \cos^2 y = 0$

Solution: We have $\frac{dy}{dx} - x^2 \cos^2 y = 0$

$$\frac{dy}{dx} = x^2 \cos^2 y$$

By separable variable, We have

$$\frac{dy}{\cos^2 y} = x^2 dx$$

$\Rightarrow$ $\sec^2 y\, dy = x^2 dx$

Integrate both sides

$$\int \sec^2 y\, dy = \int x^2 dx$$

$$\tan y = \frac{x^3}{3} + C$$

or $y = \tan^{-1}\left(\frac{x^3}{3} + C\right)$ is the general solution of differential equations.

Example 2. $\frac{dy}{dx} = 1 + x + y + xy$

Solution: We have $\frac{dy}{dx} = 1 + x + y + xy$

or $\frac{dy}{dx} = (1 + x) + y(1 + x)$

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

separating the variables,

$$\frac{dy}{1 + y} = (1 + x)dx$$

By integrating both sides

$$\int \frac{dy}{1 + y} = \int (1 + x)dx$$

$\ln(1 + y) = x + \frac{x^2}{2} + C$ is the general solution of given differential equation.

Example 3. Solve $\frac{dy}{dx} = e^{x-y}$; $y(0) = 2$

Solution: The given equation can be written as

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

By separating the variables

$$\frac{dy}{e^{-y}} = e^x dx \quad \Rightarrow \quad e^y dy = e^x dx$$

Integrating both sides

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C \qquad \ldots(i)$$

Apply $y(0) = 2$, $y = 2$ where $x = 0$, we get $e^2 = e^0 + C$

$$e^2 - 1 = C$$

Substituting the values of C in equation (i)

Hence particular solution is $e^y = e^x + e^2 - 1$.

(ii) Homogenous Differential Equation

Before going to discuss the definition of homogeneous differential equation, first we define homogeneous function.

A function $f(x, y)$ is said to be homogeneous function of degree n if it can be expressed in the form.

$$f(x, y) = x^n f\left(\frac{y}{x}\right)$$

or $\quad f(\lambda x, \lambda y) = \lambda^n f(x, y)$

For example, let $f(x, y) = \frac{x^3+y^3}{x^2-y^2}$

Replacing x *by* λx and y *by* λy

$$f(\lambda x, \lambda y) = \frac{\lambda^3 x^3 + \lambda^3 y^3}{\lambda^2 x^2 - \lambda^2 y^2} = \frac{\lambda^3(x^3 + y^3)}{\lambda^2(x^2 - y^2)} = \lambda f(x, y)$$

Thus $f(x, y)$ is homogeneous function of degree 1.

Homogeneous differential equation: A differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be homogeneous differential equation, if $f(x, y)$ and $g(x, y)$ are the homogenous functions of the same degree in x and y. For example, $\frac{dy}{dx} = \frac{x^2+y^2}{x^2-y^2}$ is homogeneous differential equation.

To solve the homogeneous differential equation, we reduce it into the separable variable form by putting

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$ where v is a new variable.

Example 1. Solve $(x^2+y^2)dx - 2xydy = 0$

Solution: We have $(x^2+y^2)dx - 2xydy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x^2\left(\frac{2y}{x}\right)} \qquad \ldots(i)$$

Thus, given differential equation is homogeneous differential equation.

Let $y = vx$

Differentiate w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x(vx)} = \frac{x^2(1+v^2)}{2x^2v} = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

By separating the variable and then integrating

$$\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$$

$$-\ln(1-v^2) = \ln x + d$$

$$\Rightarrow \quad -\ln(1-v^2) = \ln x + \ln C \qquad \text{where } d = \ln C$$

$$\ln(1-v^2)^{-1} = \ln(Cx)$$

$$\Rightarrow \quad (1-v^2)^{-1} = Cx$$

Replacing v by $\frac{y}{x}$

$$\left(1-\frac{y^2}{x^2}\right)^{-1} = Cx$$

$$\left(\frac{x^2-y^2}{x^2}\right)^{-1} = Cx$$

$$\Rightarrow \quad \frac{x^2}{x^2-y^2} = Cx$$

$$\Rightarrow \quad x^2 = Cx(x^2-y^2)$$

$$\Rightarrow \quad x = C(x^2-y^2)$$

This is the general solution of given differential equation.

Example 2. Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution: First we check the given differential equation is homogeneous or not.

Here, $f(x) = \frac{y}{x} + \tan\frac{y}{x}$

$\Rightarrow \quad f(tx, ty) = \frac{ty}{tx} + \tan\frac{ty}{tx}$

$\because \quad f(x, y) = \frac{y}{x} + \tan\frac{y}{x}$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ is homogeneous differential equation.

$\Rightarrow \quad f(tx, ty) = f(x, y)$

Let $\quad y = vx$

$\Rightarrow \quad \frac{y}{x} = v$

Differentiate w.r.t x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Given equation becomes

$$v + x\frac{dv}{dx} = v + \tan v$$

$$x\frac{dv}{dx} = \tan v$$

By separation the variables and integrating

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\int \cot v\, dv = \ln x + \ln c$$

$$\ln \sin v = \ln(c)$$

$\Rightarrow \quad \sin v = cx$

Replacing v by $\frac{y}{x}$

$$\sin\frac{y}{x} = cx$$

$$\frac{y}{x} = \sin^{-1}(cx)$$

$$y = x\sin^{-1}(cx)$$

(iii) Equations reducible to homogeneous form

The differential equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is not homogenous, but can be reduced to the homogeneous form, when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

- $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ for the differential equation $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$,

  We put $x = X + h,\ y = Y + k$

  $dx = dX$ and $dy = dY$

Then given differential equation becomes homogeneous and then we reduce it into separable variable form. We explain the method by the following example.

Example: $\frac{dy}{dx} = \frac{x-2y+2}{2x+y-1}$.

Solution: Here $\frac{a_1}{a_2} = \frac{1}{2}$ and $\frac{b_1}{b_2} = \frac{-2}{1}$ $\quad\Rightarrow\quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

So, we put $x = X + h,\ y = Y + k$

$dx = dX\ ,dy = dY$

Given equation becomes'

$$\frac{dY}{dX} = \frac{(X+h) - 2(Y+k) + 2}{2(X+h) + (Y+k) - 1} = \frac{X + h - 2Y - 2k + 2}{2X + 2h + Y + k - 1}$$

$$\frac{dY}{dX} = \frac{(X-2Y)+(h-2k+2)}{(2X+Y)+(2h+k-1)} \qquad \ldots(i)$$

To convert the equation (i) into homogeneous we assume

Let $h - 2k + 2 = 0$ and $2h + k - 1 = 0$

$\Rightarrow$ $h = 0$ and $k = 1$.

Now (i) becomes

$$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y}$$

Put $Y = VX \quad \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{X - 2VX}{2X + VX} = \frac{X(1-2V)}{X(2+V)} = \frac{1-2V}{2+V}$$

$$X\frac{dV}{dX} = \frac{1-2V}{2+V} - V$$

$$X\frac{dV}{dX} = \frac{1 - 2V - 2V - V^2}{2+V} = \frac{-(V^2 + 4V - 1)}{2+V}$$

Separating the variable and integrating

$$\int \frac{2+V}{V^2 + 4V - 1}dV = -\int \frac{dX}{X}$$

$$\frac{1}{2}\int \frac{2V+4}{V^2 + 4V - 1}dV = -\ln X + \ln C$$

$$\frac{1}{2}\ln(V^2+4V-1)=\ln\left(\frac{C}{X}\right)$$

$$\ln\sqrt{V^2+4V-1}=\ln\left(\frac{C}{X}\right)$$

Replace V by $\frac{Y}{X}$

$$\Rightarrow \quad \sqrt{\frac{Y^2}{X^2}+\frac{4Y}{X}-1}=\frac{C}{X}$$

$$\Rightarrow \quad \sqrt{\frac{Y^2+4XY-X^2}{X^2}}=\frac{C}{X}$$

$$\Rightarrow \quad \sqrt{Y^2+4XY-X^2}=C$$

$$\Rightarrow \quad Y^2+4XY-X^2=C^2$$

Replace $X=x-h=x-0=x$ and $Y=y-k=y-1$

$$(y-1)^2+4x(y-1)-(x)^2=C$$

$$y^2-2y+1+4xy-4x-x^2=C$$

This is the general solution of given differential equation.

- When $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{m}$.

then $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2}$

We put $z=a_1x+b_1y$ and $\frac{dz}{dx}=a_1+b_1\frac{dy}{dx}$

$$\Rightarrow \quad \frac{1}{b}\left(\frac{dz}{dx}-0\right)=\frac{dy}{dx}=\frac{z+c_1}{m(z)+c_2}$$

Then the given differential equation reduced to separable variable form. We explain by the following example.

Example: Solve $\frac{dy}{dx}=\frac{x-y-1}{x-y-5}$.

Solution: Here, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=1$, so, we put $z=x-y$

$$\frac{dz}{dx}=1-\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=1-\frac{dz}{dx}$$

Given equation becomes

$$1-\frac{dz}{dx}=\frac{z-1}{z-5} \quad \Rightarrow \quad -\frac{dz}{dx}=\frac{z-1}{z-5}-1$$

$$-\frac{dz}{dx}=\frac{z-1-z+5}{z-5}=\frac{4}{z-5}$$

$$\frac{dz}{dx}=-\frac{4}{(z-5)}$$

It is in separable variable form, so separating the variables and then integrating

$$\int (z-5)\,dz=\int -4dx$$

$$\frac{(z-5)^2}{2}=-4x+d$$

Replace z by $x-y$

We have $\quad (x-y-5)^2=-8x+2d$

$$x^2-2xy+y^2-10x+10y+25=4x+c$$

$$x^2-2xy+y^2-14x+16y+25=c$$

is required solution.

10.3.2 Solve real life problems related to differential equation

Example 1. If the population of a certain town doubles in 10 years, in how many years will it triple. Under the assumption that the rate of increase in population is proportional to the number of inhabitants.

Solution: Let y denote the population at time t years and y_0 at time $t=0$.

According to the given condition $\frac{dy}{dt}\propto y \Rightarrow \frac{dy}{dt}=ky$ where k is the constant.

Separating the variables and integrating

$$\int \frac{dy}{y}=\int k\,dt$$

$\ln y = kt + C$ ...(i)

Apply $\quad t=0$ and $y=y_0$, we get $\ln y_0 = C$

$\ln y = kt+\ln y_0$ ...(ii)

Apply $\quad t=10,\ y=2y_0$ (Double given)

$\ln 2y_0 = 10k+\ln y_0$

$\ln 2y_0-\ln y_0=10k \quad\Rightarrow\quad \ln\left(\frac{2y_0}{y_0}\right)=10k$

$\frac{ln2}{10}=k \quad\Rightarrow\quad k=0.06931$

Hence $\ln y = 0.06931\,t+\ln y_0$ ...(iii)

To find time t for triple population, we put $y=3y_0$

$\ln 3y_0 = 0.06931\,t+\ln y_0$

$$\ln 3y_0 - \ln y_0 = 0.06931\,t$$

$$\ln\left(\frac{3y_0}{y_0}\right) = 0.06931t \quad \Rightarrow \quad t = \frac{\ln 3}{0.06931} = 15.85$$

Hence the population will be triple in 15.85 years.

Example 2. According to Newton's law of cooling, the rate at which a substance cools in air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300K and the substance cools from 370K to 340K in 15 minutes. Find the time when the temperature will be 310K.

Solution: Let T be the temperature of the substance at the time t minutes.

Then, $$\frac{dT}{dt} \propto (T - 300)$$

$$\frac{dT}{dt} = -k(T - 300) \quad \Rightarrow \quad \frac{dT}{T - 300} = -kdt$$

Integrating with the given limit $t = 0$, when $T = 370$ and $t = 15$, when $T = 340$

$$\int_{370}^{340} \frac{dT}{T - 300} = -k \int_0^{15} dt$$

$$[\ln(T - 300)]_{370}^{340} = [-kt]_0^{15}$$

$$\ln 40 - \ln 70 = -15k$$

Now, $\Rightarrow$ $$\ln\left(\frac{7}{4}\right) = 15k$$

$\Rightarrow$ $$15k = 0.56 \quad \Rightarrow \quad k = 0.0373$$

$$\int_{370}^{310} \frac{dT}{T - 300} = -k \int_0^{t} dt$$

$$\ln(T - 300) \int_{370}^{310} = -kt$$

$$\ln 10 - \ln 70 = -kt \quad \Rightarrow \quad \ln 7 = 0.0373t$$

$\Rightarrow$ $$t = 52.2 \text{ min}$$

Example 3. A capacitor of 0.1 farads and a resistor of 10 ohm are connected in series with 100 volts battery. Assume that there is no charge and current in the circuit initially. Find the charge and current in the circuit at any time.

Solution: Potential difference at resistor

$$V = IR \qquad \text{(by ohm law)}$$

$$V = 10I$$

Potential difference at capacitor

$$Q = CV \quad \Rightarrow \quad V = \frac{Q}{0.1} = 10Q$$

By Kirchhoff's law

$$10I + 10Q = 100 \qquad \left(\because I = \frac{dQ}{dt}\right)$$

$$\frac{dQ}{dt} + Q = 10$$

$$\Rightarrow \quad \frac{dQ}{dt} = (10 - Q)$$

Separating the variables and integrating

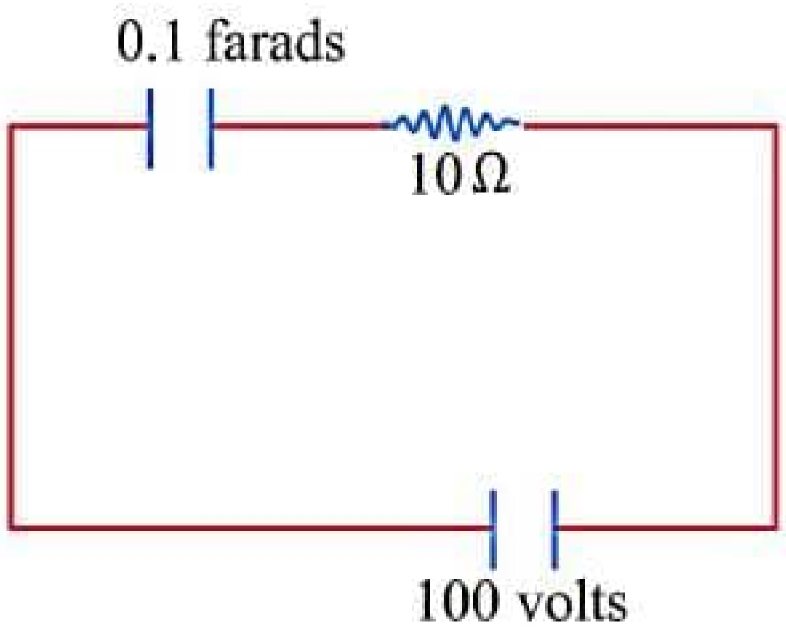


Fig. 10.1

$$\int \frac{dQ}{Q-10} = -\int dt$$

$$\ln|Q - 10| = -t + C$$

Apply $t = 0$, when $Q = 0$, we get

$$C = \ln|-10|$$

Hence $\ln(Q - 10) = -t + \ln|-10|$

$$\ln\left|\frac{Q-10}{-10}\right| = -t$$

$$\Rightarrow \quad \frac{Q-10}{-10} = e^{-t}$$

$$Q - 10 = -10e^{-t}$$

$$\Rightarrow \quad Q = 10 - 10e^{-t} = 10(1 - e^{-t})$$

Charge on the capacitor at any time.

To find current, differentiate Q w.r.t time

$$I = \frac{dQ}{dt} = 0 - 10e^{-t}(-1) = +10e^{-t}$$

$$I = 10e^{-t}$$

Example 4. A ball is thrown upward vertically with velocity 49 m/s. Find (i) the time when the body at maximum height (ii) Find the height with $t = 3$ sec (iii) Find the maximum height.

Solution: $g = -9.8\, m/s^2$

$$\frac{d^2h}{dt^2} = -9.8$$

Integrating both sides

$$\Rightarrow \quad \frac{dh}{dt} = -9.8t + c_1$$

$$V = \frac{dh}{dt} = -9.8t + c_1$$

Put $t = 0$, when $V = 49 \Rightarrow c_1 = 49$

$V = -9.8t + 49$ ...(i)

$\frac{dh}{dt} = -9.8t + 49$

Integrating both sides

$\Rightarrow h = -9.8\frac{t^2}{2} + 49t + c_2$

$t = 0, h = 0 \Rightarrow c_2 = 0$

$h = -4.9t^2 + 49t$ ...(ii)

(i) At maximum height $V = 0$ put in equation (i)

$0 = -9.8t + 49 \Rightarrow t = 5$ second

(ii) Put $t = 3$ in equation (ii) height

$h = -4.9(3)^2 + 49 = 102.9\, m$

(iii) Put $t = 5$ in equation (ii)

$h = -4.9(25) + 49(5) = 122.5\, m$

Exercise 10.2

1. Solve the following differential equation by separating the variables.

(i) $x \tan y\, dy = dx$ (ii) $x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0$

(iii) $y(1 + x)dx + x(1 + y)dy = 0$ (iv) $(1 + x^3)dy - x^2dx = 0$

(v) $xy\, dy = (y + 1)(1 - x)dx$ (vi) $\frac{dy}{dx} = 3y^2 - y^2 \sin x$

(vii) $(y^2 - 1)\frac{dy}{dx} = 4xy^2$ (viii) $x \cos^2 y\, dx + \tan y\, dy = 0$

2. Solve the following homogenous differential equation.

(i) $(x + y)dy - (x - y)dx = 0$ (ii) $(6x^2 + 2y^2)dx - (x^2 + 4xy)dy = 0$

(iii) $(x^2 + 3y^2)dx - 2xy\, dy = 0$ (iv) $(x^2 + y^2)dx - 2xy\, dy = 0$

(v) $\frac{dy}{dx} = \left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)$

3. Solve the following differential equation.

(i) $\frac{dy}{dx} = \frac{x+y+1}{x-y}$ (ii) $(2x + y + 1)\, dx + (2x + y - 1)dy = 0$

(iii) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ (iv) $\frac{dy}{dx} = \frac{2x+y-2}{2x+y+3}$

4. A body moves in a straight line, so that its velocity exceeds by 2 its distance from a fixed point of the line. If $V = 5$ m/s when $t = 0$, find the equation of the motion.

5. When the temperature of the air is 290 K a certain substance cools from 400 K to 350 K in 20 minutes. Find
   (i) the temperature after 40 minutes
   (ii) After how much time temperature is $300°C$
6. A resistor of 5 ohms and a capacitor of 0.02 farads are connected with 10 volts battery. Assume that initially charge on capacitor is 5 (coulombs). Find the charge and current in the circuit at any time.
7. The population of a certain town is directly proportional to the square root of the present population at any time. If the population initially is 20000.
   (i) How much the population after 10 years?
   (ii) After how much time the population be doubled?

10.4 Orthogonal Trajectories

10.4.1 Define and find orthogonal trajectories (rectangular coordinates) of the given family of curves.

Any family of curves $\emptyset(x, y, c) = 0$ which cuts every member of a given family of curves $f(x, y, c) = 0$ at right angles, is called an orthogonal trajectory of the given family.

For example, family of straight lines passes through origin $y = mx$ cuts every circle whose centre is at origin $x^2 + y^2 = r^2$. Hence straight right line passes through origin is orthogonal trajectory of family of circles whose centre is at origin. As shown in the figure 10.2.

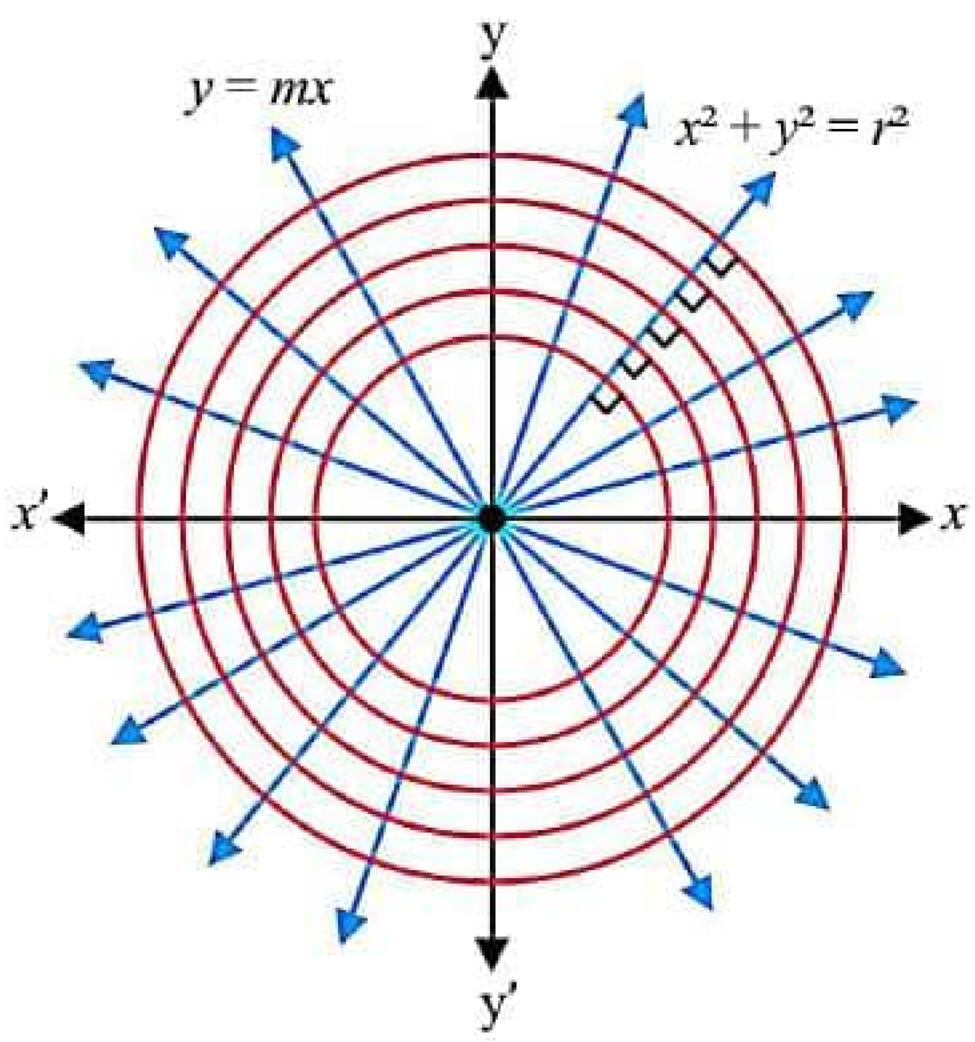


Fig. 10.2

Procedure to find orthogonal trajectories.

Step 1: Let $f(x, y, c) = 0$ be the equation of given family of curves. Where c s an arbitrary constant.

Step 2: Form the differential equations of the given family of curves.

Step 3: Substitute $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in equation obtained in step 2.

Step 4: Solve the differential equation obtained from step 3.

Example 1. Find the orthogonal trajectory of family of straight lines passing through the origin.

Solution: Family of straight line passing through the origin is $y = mx$.

Where m is an obituary constant.

$y = mx$ ...(i)

Differentiating w.r.t x

We get

$$\frac{dy}{dx} = m \quad \text{...(ii)}$$

Eliminating 'm' from equation (i) and equation (ii), we get

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{...(iii)}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$y\,dy = -x\,dx$$

Integrating on both sides

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 + x^2 = 2d$$

$$y^2 + x^2 = r^2 \qquad [\text{Assume } r^2 = 2d]$$

$$x^2 + y^2 = r^2$$

Which is orthogonal trajectory.

Example 2. Find the orthogonal trajectories of the curves $xy = c$.

Solution: The equation of the given family of curves is $xy = c$ ...(i)

Differentiating equation (i) w.r.t x,

We get

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{-y}{x} \quad \text{...(ii)}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

Integrating on both sides

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\frac{y^2}{2} - \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c$$

Which is required orthogonal trajectory.

Example 3. Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$ where a is a parameter.

Solution: Here, $x^2 + y^2 - ay = 0$ ...(i)

is the given family of curves.

Differentiating equation (i) w.r.t to x,

We get

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - a) = -2x \qquad \text{...(ii)}$$

Eliminating 'a' from equation (i) and equation (ii), we get

$$\frac{dy}{dx}\left(2y - \frac{x^2+y^2}{y}\right) = -2x \qquad \text{...(iii)}$$

$$\frac{dy}{dx}(y^2 - x^2) = -2xy$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (iii)

$$-\frac{dx}{dy} = \frac{-2xy}{y^2 - x^2}$$

$$\Rightarrow \quad -\frac{dy}{dx} = \frac{y^2-x^2}{2xy} \qquad \text{...(iv)}$$

It is homogeneous differential equation.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now equation (iv) becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2}$$
$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$
$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$
$$x\frac{dv}{dx} = -\frac{(1 + v^2)}{2v}$$

By separating the variable

$$\frac{2v}{1 + v^2} = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\ln|1 + v^2| = -\ln|x| + \ln c$$
$$\ln|1 + v^2| + \ln|x| = \ln c$$
$$\Rightarrow \quad x(1 + v^2) = c$$

Replacing v by $\frac{y}{x}$

$$x\left(1 + \frac{y^2}{x^2}\right) = c$$
$$x(x^2 + y^2) = x^2c$$
$$x^3 + y^2x - x^2c = 0$$

Which is the required equation of orthogonal trajectory.

10.4.2 Use MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories

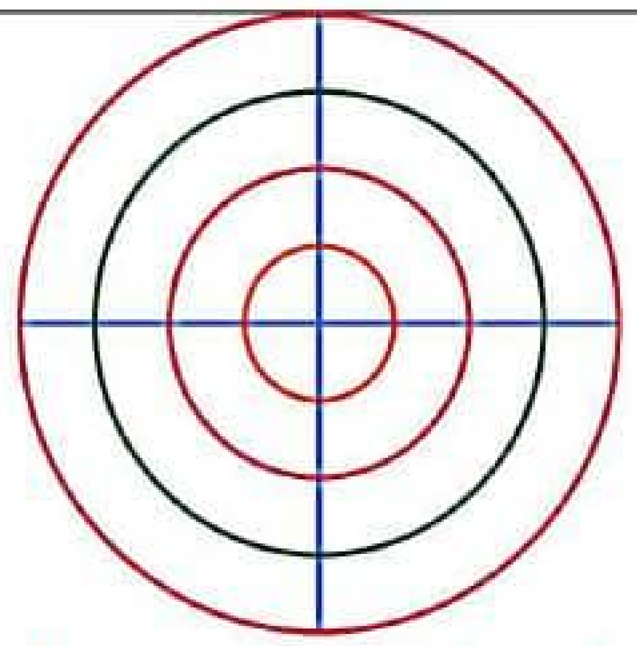
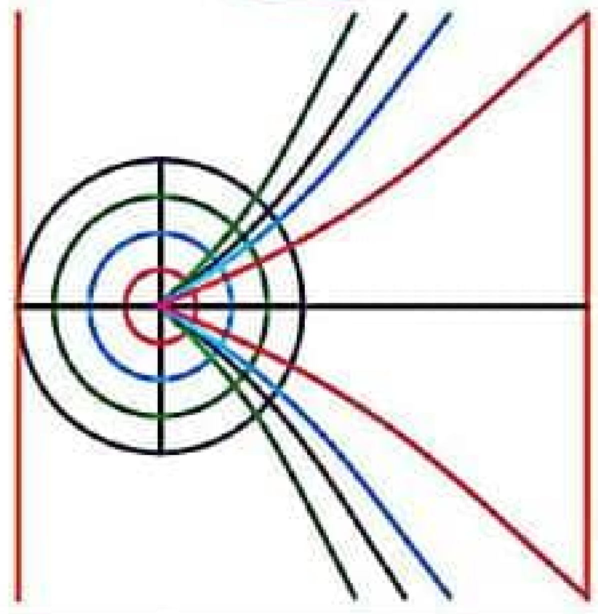
To view the graphs of given family of curves and its orthogonal trajectories following steps are to be considered:

Steps for orthogonal families of curves:

1. Differentiate the implicit function
2. Eliminate constant k of the function from the differential equation, slop of the original family
3. Form opposite reciprocal- slope of the orthogonal family
4. Separation of variables to find y the orthogonal family
5. Plot several versions of the original function and the orthogonal family

Maple Command Format	Description
$y^2 = kx^3$ $y^2 = kx^3$	To calculate or compute the orthogonal trajectories for the function $y^2 = kx^3$

Maple Command Format	Description
$\xrightarrow{\text{solve for } k}$ $\left[\left[k=\frac{y^2}{x^3}\right]\right]$	**We follow the following steps in Maple command:**
$y^2 = kx^3$	1. First, we implicitly differentiate the given function. To do this we select the function and use the differentiate option from the command pallet
$y^2 = kx^3$ $\xrightarrow{\text{implicit differentiation}}$	2. Here we choose y a dependent variable and x an independent variable in check box and get $\frac{3kx^2}{2y}$
$\frac{3kx^2}{2y}$	3. From the function we solve for k by solve option command and get $\left[\left[k=\frac{y^2}{x^3}\right]\right]$
$\frac{3\frac{y^2}{x^3}x^2}{2y}$ $\frac{3y}{2x}$	4. The value of k will be replaced by copy and paste command and get $\frac{3y}{2x}$ which is the differential function of the given function.
$y' = -\frac{2x}{3y}$	5. And write its negative reciprocal to get orthogonal prime function as $y' = -\frac{2x}{3y}$
$\frac{d}{dx}y(x) = -\frac{2x}{3y(x)}$ $\xrightarrow{\text{solve DE}}$	6. By using command pallet, from differential function command we get $\frac{d}{dx}y(x) = -\frac{2x}{3y(x)}$
$y(x) = -\frac{\sqrt{-6x^2+9c_1}}{3},$ $y(x) = \frac{\sqrt{-6x^2+9c_1}}{3}$	7. By using DE command from command pallet we get solution of this differential equation as $y(x) = -\frac{\sqrt{-6x^2+9c_1}}{3}, y(x) = \frac{\sqrt{-6x^2+9c_1}}{3}$
$y = \frac{\sqrt{-6x^2+c}}{3}$ $\xrightarrow{\text{cross multiply}}$	8. Here $9c$ in square root is just a constant it can be replaced be another constant c and by selecting one of the solution i.e., $y = \frac{\sqrt{-6x^2+c}}{3}$
$3y = \sqrt{-6x^2+c}$ $9y^2 = -6x^2 + c$	9. By cross multiply command from the pallet and square both sider we get the solution as $[[c = 6x^2 + 9y^2]]$.
$\xrightarrow{\text{solve for } c}$ $[[c = 6x^2 + 9y^2]]$ $c = 6x^2 + 9y^2$	10. Now taking different value of c i.e., 1,3,5,9 we can plot family of curves (ellipses) whose concentric center is origin. 11. Through Maple >plot 2D command from the command pallet, first we draw all the ellipses from

Maple Command Format	Description
 	following solution $$1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$$ 12. We also want to graph the original family. i.e., $$y^2 = kx^3$$ By taking different values of k i.e., 1,3,5,9 we have the following equations $$y^2 = x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$$ 13. we can draw the graphs of these function and drag them on its differential functions graph (ellipses) by using Maple which will add them in. And we'll drag and drop one at a time so we can see that they are curves, these families of curves intersect everywhere at 90 degree angles. And so, the trajectories, the paths of those functions are orthogonal.
$1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$ $1 = 6x^2 + 9y^2, 3 = 6x^2 + 9y^2, 5 = 6x^2 + 9y^2, 9 = 6x^2 + 9y^2$ $y^2 = 1x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$ $y^2 = x^3, y^2 = 3x^3, y^2 = 5x^3, y^2 = 9.x^3$	

Exercise 10.3

1. Find the orthogonal trajectory of the curves $y = ax^2$.
2. Find the orthogonal trajectories of the hyperbola $xy = c$.
3. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.
4. Find the orthogonal trajectories of the family of curves $y = \frac{x}{1+c_1x}$.
5. Find the general equation of family of curves perpendicular to the $y = c_1 \sin x$.
6. Find the general equation of family of curves perpendicular to the $x^{\frac{1}{3}} + y^{\frac{1}{3}} = c$.

Review Exercise 10

1. Tick the correct answer.

(i) The order and degree of the differential equation $1 + \frac{d^2y}{dx^2} = x\frac{dy}{dx}$ is ________.

(a) order 2, degree 2 (b) order 2, degree 1

(c) order 1, degree 2 (d) order 1, degree 1

(ii) The degree of differential equation $\left(\frac{dy}{dx}\right)^4 + 3\frac{d^2y}{dx^2}$ is ________.

(a) 2 (b) 4 (c) 1 (d) 3

(iii) The order and degree of differential equation $\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt{\frac{dy}{dx}}$ is ________.

(a) order 3, degree 4 (b) order 4, degree 3
(c) order 2, degree 1 (d) order 1, degree 2

(iv) The degree of differential equation $y = x\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)$ is ________.

(a) 1 (b) 2 (c) 3 (d) 4

(v) The order and degree of differential equation $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$ is ________.

(a) 2, 2 (b) 2, 3 (c) 3, 2 (d) 2, 1

(vi) Differential equation $xdy - ydx = 0, y(1) = 2$ has a solution given by y. Then $y(-1)$ is ________.

(a) -1 (b) -2 (c) 2 (d) 1

(vii) The solution of differential equation $\frac{dy}{dx} + y^2 = 0$ is ________.

(a) $y = ce^n$ (b) $y = \frac{1}{x+c}$
(c) $y = -\frac{x^3}{3} + c$ (d) $y = \frac{x^2}{3} + c$

(viii) The general solution of the differential equation $9y\frac{dy}{dx} + 4x = 0$ is ________.

(a) $4x^2 + 9y^2 = c$ (b) $\frac{x^2}{9} - \frac{y^2}{4} = c$
(c) $4x^2 + y^2 = c$ (d) $9x^2 - 4y^2 = 0$

2. Show that $y = Ae^{3x} + Be^{4x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0.$

3. Solve the following differential equation

(i) $\cos(x + y)\,dy = dx$ (ii) $x^2\frac{dy}{dx} = x^2 + xy + y^2$

(iii) $(xy + y^2)dx = (x^2 - xy)dy$ (iv) $x\frac{dy}{dx} = y + x\tan\frac{y}{x}$

(v) $(2x + 3y - 5)dy + (3x + 2y + 1)dx = 0$

(vi) $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$

(vii) $\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y}{x}\right)$ (viii) $\frac{dy}{dx} = e^{x+y}$

4. Given $y = (A + Bx)\,e^{-2x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0,\ y(0) = 2,\ y'(0) = 5$. Find the particular solution.

5. Form / obtain the differential equation by eliminating arbitrary constants from the given relation

(i) $y = \frac{9}{x} + b$ (ii) $y = e^x(a\cos x + b\sin x)$

(iii) $y = \ln\cos(x - a) + b$ (iv) $y = A\cos(\ln x) + B\sin(\ln x)$

6. A body moves along a straight line, its acceleration after t sec is given by $\frac{1}{\sqrt{t}}$.

At $t = 9$ sec, $V = 25$ m/s. Find the velocity at any time and at $t = 20$ sec.

7. Find the orthogonal trajectories of the family of the curve $3x + 4y = c$.

8. Find the general equation of the family of the curves perpendicular to the $y = \ln(\tan x + c_1)$.

Unit 11 Partial Differentiation

11.1 Differentiation of function of two variables

We have already studied the differentiation of function of one variable. Now, in this section, we will focus on differentiation of function of two variables.

11.1.1 Define a function of two variables

If a quantity z has unique and finite value for every pair of values x and y, then z is called function of two independent variables x and y.

i.e., $$z = f(x, y)$$

Here, z possesses unique and finite value for each ordered pair $(x, y) \in \mathbb{R}^2$.

For example, $f(x, y) = x^2 + xy + y^2$ is a function of two variables, because for different values of x and y, f has a unique and finite value.

11.1.2 Define partial derivative

The concept of partial derivative arises when function is of two or more variables.

Definition:

Let f is the function of two variables x and y, denoted by $f(x, y)$, then partial derivative of f with respect to x is the ordinary derivative of $f(x, y)$ with respect to x by taking y as a constant. It is denoted as $\frac{\partial f}{\partial x}$ or f_x. Similarly, partial derivative of $f(x, y)$ with respect to y can be defined, and is denoted by $\frac{\partial f}{\partial y}$ or f_y.

11.1.3 Find partial derivatives of a function of two variables

Example 1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, given that

(i) $f(x, y) = x^2 + xy + y^2$ (ii) $f(x, y) = ye^x$

(iii) $f(x, y) = \ln y, y > 0$

(i) As $f(x, y) = x^2 + xy + y^2$

Differentiating f partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2)$$

$$\frac{\partial f}{\partial x} = 2x + y(1) + 0$$

$$\frac{\partial f}{\partial x} = 2x + y$$

Similarly, differentiating f partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

$$\frac{\partial f}{\partial y} = 0 + x(1) + 2y$$

$$\frac{\partial f}{\partial y} = x + 2y$$

(ii) $\boldsymbol{f(x, y) = ye^x}$

Differentiating f partially with respect to x

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(ye^x)$$

$$\frac{\partial f}{\partial x} = y\left(\frac{\partial e^x}{\partial x}\right)$$

$$\frac{\partial f}{\partial x} = ye^x$$

Similarly, differentiating f partially with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(ye^x)$$

$$\frac{\partial f}{\partial y} = e^x(1)$$

$$\frac{\partial f}{\partial y} = e^x$$

(iii) $\boldsymbol{f(x, y) = \ln y\,, y > 0}$

Differentiating partially with respect to x, we get

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\ln y)$$

$$\frac{\partial f}{\partial x} = \ln y \frac{\partial}{\partial x}(1)$$

$$\frac{\partial f}{\partial x} = \ln y \times 0$$

$$\frac{\partial f}{\partial x} = 0$$

Similarly, differentiating partially with respect to y, we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\ln y)$$

$$\frac{\partial f}{\partial y} = \frac{1}{y}$$

Example 2. Find the partial derivative of the area of triangle by base as well as height of the triangle.

Solution: The area of triangle is defined as $A = \frac{1}{2}bh$

Here b and h are base and height of the triangle respectively.

Now, partial derivative of Area A with respect to base b is

$$A = \frac{1}{2}bh$$

Differentiating A partially w.r.t b, we get

$$\frac{\partial A}{\partial b} = \frac{\partial}{\partial b}\left(\frac{1}{2}bh\right)$$

$$\frac{\partial A}{\partial b} = \frac{1}{2}h\,(1) \qquad \text{(here } h \text{ is treated as constant coefficient)}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2}h$$

Similarly, differentiating A partially w.r.t h, we get

$$\frac{1}{2}bh$$

$$\frac{\partial A}{\partial h} = \frac{\partial}{\partial h}\left(\frac{1}{2}bh\right)$$

$$\frac{\partial A}{\partial h} = \frac{1}{2}b(1) = \frac{b}{2} \qquad \text{(here } b \text{ is treated as constant coefficient)}$$

11.2 Euler's Theorem

Euler's theorem is one of the most important theorems of calculus, which contains homogeneous function and its partial derivative.

11.2.1 Define a homogeneous function of degree n

Definition: A function $f(x,y)$ is said to be a homogeneous function of degree n if it can be written in the form of $\quad f(tx,ty) = t^n f(x,y)$

or $\quad f(x,y) = x^n f\left(\frac{y}{x}\right)$

Example 1. Show that the polynomial function in two variables $p(x,y) = x^3 + x^2y + xy^2 + y^3$ is the homogeneous function of degree 3.

Solution: As $p(x,y) = x^3 + x^2y + xy^2 + y^3$

By taking highest power of x as common

$$p(x,y) = x^3\left[1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}\right]$$

$$p(x,y) = x^3\left[\left(\frac{y}{x}\right)^0 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right]$$

$$p(x,y) = x^3\, p\left[\left(\frac{y}{x}\right)\right]$$

Hence $p(x,y)$ is the homogeneous function of degree 3.

Example 2. Show that the function $f(x,y) = \frac{x^4+y^4}{x-y}$ is homogeneous function of degree 3.

Solution: Replacing x by tx and y by ty

$$f(tx,ty) = \frac{(tx)^4 + (ty)^4}{tx - ty}$$

$$f(tx,ty) = \frac{t^4x^4 + t^4y^4}{t(x-y)}$$

$$= \frac{t^4(x^4 + y^4)}{t(x-y)}$$

$$f(tx,ty) = t^3\left(\frac{x^4 + y^4}{x - y}\right)$$

$$f(tx,ty) = t^3 f(x,y)$$

$\therefore$ $f(x,y) = \frac{x^4+y^4}{x-y}$ is homogeneous function of degree 3. Hence shown.

Alternatively, $f(x,y) = \frac{x^4+y^4}{x-y} = \frac{x^4\left(1+\frac{y^4}{x^4}\right)}{x\left(1-\frac{y}{x}\right)}$

$$= x^3\left(\frac{1+\left(\frac{y}{x}\right)^4}{1-\left(\frac{y}{x}\right)}\right)$$

$$f(x,y) = x^3 f\left(\frac{y}{x}\right)$$

This shows that $f(x,y) = \frac{x^4+y^4}{x-y}$ is the homogeneous function degree 3.

Example 3. Show that the function $f(x,y) = \sin\left(\frac{x^2+y^2}{x-y}\right)$ is not a homogeneous function.

Solution: Here $f(x,y) = \sin\left(\frac{x^2+y^2}{x-y}\right)$

Replacing x by tx and y by ty , we get

$$f(tx,ty) = \sin\left(\frac{(tx)^2 + (ty)^2}{(tx) - (ty)}\right)$$

$$= \sin\left(\frac{t^2x^2 + t^2y^2}{tx - ty}\right)$$

$$= \sin\left(\frac{t^2(x^2 + y^2)}{t(x - y)}\right)$$

$$f(tx, ty) = \sin\left(t\left(\frac{x^2 + y^2}{x - y}\right)\right)$$

$$\because \sin\left(t\left(\frac{x^2 + y^2}{x - y}\right)\right) \neq t\sin\left(\frac{x^2 + y^2}{x - y}\right)$$

$$\therefore \quad f(tx, ty) \neq t\, f(x, y)$$

Hence, $f(x, y)$ is not homogeneous a function.

Example 4. Show that the function $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$ is homogeneous of degree $-\frac{3}{2}$.

Solution: As $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$

$$= \frac{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)}{x^2\left(1 - \frac{y^2}{x^2}\right)}$$

$$= x^{-\frac{3}{2}} \frac{\left(1 + \left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{\left(1 - \left(\frac{y}{x}\right)^2\right)} \qquad \left[\because f(x, y) = x^n f\left(\frac{y}{x}\right)\right]$$

$$f(x, y) = x^{-\frac{3}{2}} f\left(\frac{y}{x}\right)$$

Hence $f(x, y) = \frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$ is the homogeneous function of degree $-\frac{3}{2}$.

11.2.2 State and prove Euler's theorem on homogeneous functions

Let $z = f(x, y)$ is a homogeneous function of degree n, then by Euler's theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Proof: It is given that $z = f(x, y)$ is the homogeneous function of degree n. So, it can be written as

$$z = f(x, y) = x^n f\left(\frac{y}{x}\right) \qquad \text{... (i)}$$

From the statement of Euler's theorem, we need the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Differentiating partially (i) with respect to x, we get

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^n f\left(\frac{y}{x}\right)\right]$$

By applying product rule of derivative, we get

$$\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n) + x^n \frac{\partial}{\partial x} f\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} y f'\left(\frac{y}{x}\right)$$

Multiplying both sides by x, we get

$$x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \text{... (ii)}$$

Similarly, differentiating (i) partially with respect to y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^n f\left(\frac{y}{x}\right)\right)$$

$$\frac{\partial z}{\partial y} = x^n \frac{\partial}{\partial y} f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^{n-1} f'\left(\frac{y}{x}\right)$$

Multiplying both sides by y, we get

$$y\frac{\partial z}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \text{... (iii)}$$

By adding equations (ii) and (iii), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left[nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right)\right] + x^{n-1} y f'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nf(x,y) \qquad \left[\because f(x,y) = x^n f\left(\frac{y}{x}\right)\right]$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Hence proved.

11.2.3 Verify Euler's theorem for homogeneous functions of different degrees (simple cases)

Example 1. Let $z = \frac{x^2y^2}{x^2+y^2}$ then verify Euler's theorem.

As, $z = \frac{x^2y^2}{x^2+y^2}$ is homogeneous function of degree 2. Then by Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \text{... (i)}$$

To verify this, we find partial derivatives of z.

$$z = \frac{x^2y^2}{x^2+y^2}$$

Differentiating partially with respect to x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2y^2}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(x^2y^2) - (x^2y^2)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{(x^2+y^2)(2xy^2) - (x^2y^2)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^3y^2 + 2xy^4 - 2x^3y^2}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} = \frac{2x^2y^4}{(x^2+y^2)^2} \qquad \text{... (ii)}$$

Similarly, differentiating partially with respect to y, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2)\frac{\partial}{\partial y}(x^2y^2) - (x^2y^2)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2+y^2)(2x^2y) - (x^2y^2)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2x^4y + 2x^2y^3 - 2x^2y^3}{(x^2+y^2)^2}$$

$$y\frac{\partial z}{\partial y} = \frac{2x^4y^2}{(x^2+y^2)^2} \qquad \text{... (iii)}$$

By adding equations (ii) and (iii), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^4y^2}{(x^2+y^2)^2} + \frac{2x^2y^4}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2y^2(x^2+y^2)}{(x^2+y^2)^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2x^2y^2}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad \left[\because z = \frac{x^2y^2}{x^2+y^2}\right]$$

Hence verified.

Example 2. Given that $p(x,y) = ax^2 + bxy + cy^2$ be the homogeneous function of degree 2. Then verify Euler's theorem for it.

Proof: As $p(x,y) = ax^2 + bxy + cy^2$ is the homogeneous function of degree 2. Then by Euler's theorem.

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} = 2p \qquad \text{... (i)}$$

To verity this, first we find partial derivatives of $p(x,y)$.

$$p(x,y) = ax^2 + bxy + cy^2$$

$$\frac{\partial p}{\partial x} = 2ax + by$$

Multiplying both sides by x, we get

$$x\frac{\partial p}{\partial x} = 2ax^2 + bxy \qquad \text{... (ii)}$$

Similarly,

$$\frac{\partial p}{\partial y} = bx + 2cy$$

Multiplying both sides by y, we get

$$y\frac{\partial p}{\partial y} = bxy + 2cy^2 \qquad \text{... (iii)}$$

By adding equation (ii) and equation (iii)

We get,

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} = 2ax^2 + bxy + bxy + 2cy^2$$

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=2(ax^2+bxy+cy^2)$$

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=2p(x,y)$$

Hence verified.

Example 3. Verify Euler's theorem for the function $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$.

Solution: Let us check the homogeneity and degree of the function.

Here $f(x,y)=z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

Replacing x by tx and y by ty, we get

$$f(tx,ty)=\sin^{-1}\left(\frac{tx}{ty}\right)+\tan^{-1}\left(\frac{ty}{tx}\right)$$

$$f(tx,ty)=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$$

$$f(tx,ty)=t^0\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

Hence $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ is the homogeneous function of degree 0.

$\therefore$ By Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.z=0 \qquad \text{... (i)}$$

To verify Euler's theorem, we find the partial derivatives of $z=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ w.r.t their independent variables.

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y}+\frac{x^2}{x^2+y^2}\cdot\left(\frac{-y}{x^2}\right)$$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

Multiplying both sides by x, we get

$$x\frac{\partial z}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2} \qquad \text{... (ii)}$$

Similarly,

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

Multiplying both sides by y, we get

$$y\frac{\partial z}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{... (iii)}$$

By adding equations (ii) and (iii), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Hence verified.

11.2.4 Use MAPLE command diff to find partial derivative

The format of diff command to partial derivative of a function in MAPLE is as under:

$> diff(f, x, y)$ is equivalent to the command $\frac{\partial}{\partial x} f$ in Maple version 2022.

Where,

f stands for function whose partial derivative is to be evaluated

X, Y stands for the variable x and y, the partial derivative with respect to x or y.

$\frac{\partial}{\partial x}$ means 1st order partial derivative with respect to variable x

Note: All above operators should be taken from the Maple calculus pallet.

Use MAPLE command **diff** or $\left(\frac{\partial}{\partial x} f\right)$ to differentiate a function:

Partial Derivative of functions:

$> f := (x, y) \rightarrow (x^2y + 5xy + xy^2)$

$f := (x, y) \rightarrow x^2y + 5xy + xy^2$

$> diff(f(x, y), x)$

$2xy + y^2 + 5y$

$> diff(f(x, y), y)$

$x^2 + 2xy + 5y$

$> diff(f(x, y), x, y)$

$2x + 2y + 5$

$> f := (x, y) \rightarrow (x + \ln(xy) + 2x\sin(y)^2)$

$f := (x, y) \rightarrow x + \ln(yx) + 2x\sin y^2$

$> diff(f(x, y), x)$

$1 + \frac{1}{x} + 2\sin y^2$

$> diff(f(x, y), y)$

Partial Derivative of functions:

> $diff(f(x,y),y,x)$

$$2x+2y+5$$

> $diff(f(x,y),x,x)$

$$2y$$

> $diff(f(x,y),y,y)$

$$2x$$

$$\frac{1}{y}+4x\sin y$$

> $diff(f(x,y),x,y)$

$$4\sin y$$

> $diff(f(x,y),x,x)$

$$-\frac{1}{x^2}$$

> $diff(f(x,y),y,y)$

$$-\frac{1}{y^2}+4\sin x$$

> $f := (x,y) \longrightarrow (x+y+ye^x)$

$$f := (x,y) \longrightarrow x+y+ye^x$$

> $diff(f(x,y),x)$

$$1+ye^x$$

> $diff(f(x,y),y)$

$$1+e^x$$

> $diff(f(x,y),x,x)$

$$ye^x$$

> $diff(f(x,y),y,y)$

$$0$$

> $f := (x,y) \longrightarrow (\ln(x+1)+y+ye^x)$

$$f := (x,y) \longrightarrow \ln(x+1)+y+ye^x$$

> $diff(f(x,y),x)$

$$\frac{1}{1+x}+ye^x$$

> $diff(f(x,y),y)$

$$1+e^x$$

> $diff(f(x,y),x,x)$

$$-\frac{1}{(1+x)^2}+ye^x$$

> $diff(f(x,y),y,y)$

$$0$$

Exercise 11

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when $f(x,y)$ is given by
   (i) $f(x,y)=3x^3+y^2-6x+2y-7$
   (ii) $f(x,y)=x^2+xy-y^2-2x-2y-8$ (iii) $f(x,y)=\sin(x+y)$
   (iv) $f(x,y)=e^x\cos y$ (v) $f(x,y)=\frac{1}{2}\ln(x^2+y^2)$
2. The volume of the cone is given by formula $V=\frac{1}{3}\pi r^2 h$. Differentiate V with respect to their independent variables.
3. Check whether the following functions are homogeneous or not. Find the degree in case of homogeneous function.
   (i) $f(x,y)=\frac{x^3-5x^2y+7xy^2+y^3}{xy^2}$ (ii) $f(x,y)=\tan\left(\frac{x+y}{y^2}\right)$

(iii) $f(x,y) = x^3 + 3x^2y + 2y^2x + y^3$ (iv) $f(x,y) = \cos^{-1}\left(\frac{x^2-y^2}{xy}\right)$

(v) $f(x,y) = \frac{x^2-xy+y^2}{x+y^2}$ (vi) $f(x,y) = \sqrt{x^8 - 3x^2y^6}$

(vii) $f(x,y) = x^3 \sin\left(\frac{y^2}{x}\right)$ (viii) $f(x,y) = \ln\left(\frac{x^2+y^2}{x+y}\right)$

4. Verify Euler's theorem for the following homogeneous function.

(i) $f(x,y) = xy + y^2$ (ii) $f(x,y) = \cos\left(\frac{x}{y}\right)$

(iii) $f(x,y) = \sqrt{xy} - x$ (iv) $f(x,y) = \ln\left(\frac{x+y}{y}\right)$

5. If $u = x^2(y-x) + y^2(x-y)$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -2(x-y)^2$.

6. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

7. Use MAPLE command >**diff** or $\left(\frac{\partial}{\partial x} f\right)$ to partial differentiate with respect to x and y of the following functions:

(i) $f(x,y) = x^2y + xy + xy^2$ (ii) $f(x,y) = y + x\cos(y)$

(iii) $f(x,y) = \frac{x+\sqrt{x}}{y+\sqrt{y}}$

Review Exercise 11

1. Multiple choice questions (MCQs)

(i) Given that $f(x,y) = e^{xy}$ then $\frac{f_x}{f_y} =$ ____________.

(a) $\frac{x}{y}$ (b) 1 (c) $\frac{y}{x}$ (d) $\frac{y+x\,f'(x,y)}{x+y\,f'(x,y)}$

(ii) Surface area of a cube is a function of __________ variables.

(a) 1 (b) 2 (c) 3 (d) 4

(iii) Given that $g(x,y) = \cos\left(\frac{x}{y}\right)$ then $\frac{g_y}{g_x} =$ ___________.

(a) $-\frac{x}{y}$ (b) $\frac{x}{y}$ (c) $-\frac{x}{y^3}$ (d) $-\frac{y}{x}$

(iv) A function $\tan\left(\frac{2x}{3y}\right)$ is a homogeneous function of degree ___________.

(a) undefined (b) $\frac{2}{3}$ (c) 1 (d) 0

(v) The perimeter of rectangle is given by a function $P(x,y) = 2(x+y)$, where x and y are respectively its length and breadth. Then sum of partial derivatives w.r.t their independent variables is ___________.

(a) $2x$ (b) $2y$ (c) $2(x+y)$ (d) 4

(vi) Given that $z = f(x,y)$ is a homogeneous function of degree 0 then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$ __.

(a) $(x+y)\,f'(x,y)$ (b) $x+y$ (c) 0 (d) $f(x,y)$

(vii) The area of trapezium is a function of _______________ variables.

(a) 1 (b) 2 (c) 3 (d) 4

(viii) Given $w = f(u,v)$ is a homogeneous function of degree $\frac{2}{3}$ then $u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} =$ ____.

(a) $(u+v)f'(u,v)$ (b) 0 (c) $\frac{2}{3}$ (d) $\frac{2}{3}w$

(ix) Given that $z = y\left(\frac{u}{v}\right)$ is a homogeneous function of degree 0 then $v\frac{\partial z}{\partial v} =$ _______.

(a) $u\frac{\partial z}{\partial u}$ (b) 0 (c) $-u\frac{\partial z}{\partial u}$ (d) -1

(x) Let $f(x,y)$ and $g(x,y)$ are homogeneous functions of degrees 2 and 3 respectively, then degree of homogeneous functions $\frac{f(x,y)}{g(x,y)}$ is ____________.

(a) 6 (b) 1 (c) $\frac{2}{3}$ (d) -1

2. Let $f(x,y) = xy$ and $g(x,y) = \frac{1}{2}xy$ be the homogeneous functions for the areas of rectangle and triangle respectively, where x and y are their independent variables. Are $f+g, f-g, fg$ and $\frac{f}{g}$ homogenous? If yes, what are their degrees?

3. Verify Euler's theorem for the function $z = \sqrt{x^2+y^2}$

4. Given that $z = g(x,y)$ is a homogeneous function of degree 3 then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$.

5. Given that $y = f(u,v)$ is a homogeneous function of degree $-\frac{3}{2}$ then show that $u\frac{\partial y}{\partial u} + v\frac{\partial y}{\partial v} = -\frac{3}{2}y$.

6. Given that $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

7. Given that $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

Introduction to Numerical Methods

Unit 12

12.1 Numerical Solution of Non-linear Equations

12.1.1 Describe importance of numerical methods

Till now, all the methods we have learnt to solve the non-linear equation and finding of the derivative or integration of functions are analytical methods.

When analytic approaches are failed to find the solution of a non-linear equation or require too many tedious computations then, mathematicians used numerical methods to compute approximate solution. Therefore, numerical methods have great importance in the field of mathematics.

12.1.2 Explain the basic principles of solving a non-linear equation in one variable

The basic principle of solving a non-linear equation is to find the interval (values of a and b) for a function $f(x)$, where $f(a)$ and $f(b)$ are of opposite signs such that $f(a).f(b) < 0$, then the root of $f(x) = 0$ lies in the interval $[a, b]$.

For example, $f(x) = x^3 - 2x - 5$, put the values of $x = 0,1,2,3$ in $f(x)$, we get

$$f(0) = (0)^3 - 2(0) - 5 = -5$$
$$f(1) = (1)^3 - 2(1) - 5 = -6$$
$$f(2) = (2)^3 - 2(2) - 5 = -1$$
$$f(3) = (2)^3 - 2(3) - 5 = 16$$

Here $f(2) = -1 < 0(-ve)$ and $f(3) = 16 > 0(+ve)$,
such that $f(2).f(3) = (-1).(16) = -16 < 0$,
Now, root lies in the interval $[a, b] = [2, 3]$.

12.1.3 Calculate the real roots of a non-linear equation in one variable by

- bisection method
- regula-falsi method
- Newton-Raphson method

- **Bisection Method**

The bisection technique is a root-finding method which repeatedly bisects an interval and then selects a sub-interval in which a root must lie for further processing. It is a very easy and reliable procedure.

Algorithm of a Bisection method

If $f(x)$ is a continuous function over an interval, then to find the root of $f(x) = 0$ by bisection method, following steps are taken.

Step 1. Choose two approximations a and $b(b > a)$ such that $f(a).f(b) < 0$.

Step 2. Evaluate the midpoint c of an interval $[a, b]$ given by $c = \frac{a+b}{2}$.

Step 3. Now there are three possibilities:

(i) If $f(c) = 0$, , then c is a root of $f(x) = 0$.

(ii) If $f(c) < 0$, and $f(a)f(c) < 0$ then root lies in the $[a, c]$ else if $f(b)f(c) < 0$ then root lies in the $[c, b]$

Step 4. Continue the process till the root is found to the desired accuracy, that is two decimal places or three decimal places or four decimal places etc.

Example 1. Use Bisection method to find a root of an equation $x^2 - 3 = 0$ up to four iteration.

Solution:

Let $f(x) = x^2 - 3 = 0$

Taking $a = 1$ and $b = 2$.

Here $f(1) = 1 - 3 = -2 < 0$ and $f(2) = 4 - 3 = 1 > 0$

Since $f(1)f(2) < 0$

Therefore, root lies between 1 and 2

1st iteration

Now,

$$c = \frac{1+2}{2} = 1.5$$

$$f(c) = f(1.5) = (1.5)^2 - 3 = -0.75 < 0$$

Since $f(2)f(1.5) < 0$

therefore, root lies between 1.5 and 2

2nd iteration

Taking $a = 1.5$ and $b = 2$

$$c = \frac{(1.5 + 2)}{2} = 1.75$$

$$f(c) = f(1.75) = (1.752)^2 - 3 = 0.062 > 0$$

Since $f(1.5)f(1.75) < 0$

therefore, root lies between 1.5 and 1.75

3rd iteration

Taking $a = 1.5$ and $b = 1.75$

$$c = \frac{(1.5 + 1.75)}{2} = 1.625$$

$$f(c) = f(1.625) = (1.6252)^2 - 3 = -0.359 < 0$$

Since $f(1.75)f(1.625) < 0$

therefore, root lies between 1.625 and 1.75

4th iteration

Taking $a = 1.625$ and $b = 1.75$

$$c = \frac{(1.625 + 1.75)}{2} = 1.688$$

$$f(c) = f(1.688) = (1.6882)^2 - 3 = -0.152 < 0$$

Hence 1.688 is the approximate root of $x^2 - 3 = 0$ after four iterations.

Example 2. Find a root of an equation $f(x) = x^3 + 2x^2 + x - 1$, using Bisection method correct to two decimal places.

Solution:

Here $f(x) = x^3 + 2x^2 + x - 1 = 0$

Find the value of $f(x)$ at $x = 0,1$

$f(0) = (0)^3 + 2(0)^2 + (0) - 1 = -1$

$f(1) = (1)^3 + 2(1)^2 + (1) - 1 = 3$

Here $f(0) = -1 < 0$ and $f(1) = 3 > 0$

Since $f(0)f(1) < 0$

therefore, root lies between 0 and 1

1st iteration

Taking $a = 0$ and $b = 1$

$$c = \frac{0 + 1}{2} = 0.5$$

$$f(0.5) = (0.5)^3 + 2(0.5)^2 + (0.5) - 1 = 0.125 > 0$$

Here $f(0) = -1 < 0$ and $f(0.5) = 0.125 > 0$

Since $f(0)f(0.5) < 0$

therefore, root lies between 0 and 0.5

2nd iteration

Taking $a = 0$ and $b = 0.5$

$$c = \frac{0 + 0.5}{2} = 0.25$$

$$f(0.25) = (0.25)^3 + 2(0.25)^2 + (0.25) - 1 = -0.6094 < 0$$

Here $f(0.25) = -0.6094 < 0$ and $f(0.5) = 0.125 > 0$

Since $f(0.25)f(0.5) < 0$

therefore, root lies between 0.25 and 0.5

3rd iteration

Taking $a = 0.25$ and $b = 0.5$

$$c = \frac{0.25 + 0.5}{2} = 0.375$$

$$f(0.375) = (0.375)^3 + 2(0.375)^2 + (0.375) - 1 = -0.291 < 0$$

$$f(0.375) = -0.291 < 0 \text{ and } f(0.5) = 0.125 > 0$$

Since $f(0.375)f(0.5) < 0$

therefore, root lies between 0.375 and 0.5

4th iteration

Taking $a = 0.375$ and $b = 0.5$

$$c = \frac{0.375 + 0.5}{2} = 0.4375$$

$$f(0.4375) = (0.4375)^3 + 2(0.4375)^2 + (0.4375) - 1 = -0.0959 < 0$$

Since $f(0.4375) < 0$ and $f(0.5) > 0$.

therefore, root lie in the interval.

5th iteration

Taking $a = 0.4375$ and $b = 0.5$

$$c = \frac{0.4375 + 0.5}{2}$$

$$c = 0.4688$$

$$f(0.4688) = (0.4688)^3 + 2(0.4688)^2 + 0.4688 - 1$$

$$f(0.4688) = 0.112 > 0$$

Since $f(0.4688)f(0.4375) < 0$

therefore, root lie between 0.4375 and 0.4688.

6th iteration

Taking $a = 0.4375$ and $b = 0.4688$

$$c = \frac{0.4375 + 0.4688}{2}$$

$$c = 0.4531$$

$$f(0.4531) = (0.4531)^3 + 2(0.4531)^2 + 0.4531 - 1$$

$$f(0.4531) = -0.0432 < 0$$

Since $f(0.4531)f(0.4688) < 0$

therefore, root lie between 0.4531 and 0.4688.

7th iteration

Taking $a = 0.4531$ and $b = 0.4688$

$$c = \frac{0.4531 + 0.4688}{2} = 0.4609$$

$$f(0.4609) = (0.4609)^3 + 2(0.4609)^2 + 0.4609 - 1$$

$$f(0.4609) = -0.0162 < 0$$

Since $f(0.4609)f(0.4688) < 0$

therefore, root lie between the 0.4609 and 0.4688.

8th iteration

Taking $a = 0.4609$ and $b = 0.4688$

$$c = \frac{0.4609 + 0.4688}{2} = 0.4648$$

$$f(0.4648) = (0.4648)^3 + 2(0.4648)^2 + 0.4648 - 1$$

$$f(0.4648) = -0.0026$$

Hence, we obtained accuracy up to two decimal places. Therefore 0.4648 is required approximate root.

12.1.4 Use MAPLE command fsolve to find numerical solution of an equation and demonstrate through examples

The fsolve command is the numeric equivalent of solve. The fsolve command finds the roots of the equation(s), producing approximate (floating-point) solutions.

Examples:

$> polynomial := 3x^4 - 16x^3 - 3x^2 + 13x + 16$

$> fsolve(polynomial)$

$$1.324717957, 5.333333333$$

$> polynomial := x^6 - x - 1$

$> fsolve(polynomial)$

$$-0.7780895987, 1.134724138$$

$> fsolve(2x + y = 17, x^2 - y^2 = 20, x, y)$

$$\{x = 16.37758198, y = -15.75516397\}$$

$> f := sin(x + y) - exp(x) * y = 0$

$> g := x^2 - y = 2$

$> fsolve(f, g, x = -1..1, y = -2..0)$

$$\{x = -0.6687012050, y = -1.552838698\}$$

$> \text{fsolve}(\cos(x) - x = 0, x);$

$$\{x = .7390851332\}$$

Exercise 12.1

1. Use Bisection method to find a real root of the following equations.
   - (i) $f(x) = 2x^3 - 2x - 5, [1,2]$ up to three iterations
   - (ii) $f(x) = x^3 - 2x - 5, [1.5, 2.5]$ up to four iterations
   - (iii) $f(x) = x^3 - x + 1, [-2, -1]$ up to five iterations
   - (iv) $f(x) = cos\, x, [1, 2]$ up to one decimal place (five iterations)
   - (v) $f(x) = 3x - e^x, [0, 1]$ up to three decimal places (eleven iterations)
   - (vi) $f(x) = 3x - \sqrt{1 + sin\, x}, [0,1]$ up to three decimal places (thirteen iterations)
2. Write MAPLE command fsolve to find numerical solution the following:
   - (i) polynomial :$3x^4y^2 = 17, x^2y - 5xy^2 - 2y = 1$
   - (ii) polynomial :$3x^3 - 27x + 3$
   - (iii) polynomial :$3x^3 + 9x + 3$
   - (iv) polynomial :$2x^3 + 4x + 2$

- **Regula Falsi method**

This approach is also known as the false position method. It is an iterative method for determining the real root of a nonlinear equation $f(x) = 0$. This method gives a better approximation for the roots of the equation than bisection method.

- **Algorithm of Regula Falsi method:**

Let $f(x)$ is a continuous function over the interval, to find the approximate root of $f(x) = 0$ by Regula Falsi Method following steps are taken.

Step 1. Find points a and b such that $a < b$ and $f(a).f(b) < 0$.

Step 2. Take the interval $[a, b]$ and find next value using

formula: $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

Step 3. If $f(x_1) = 0$ then x_1 is an exact root.

else if, $f(x_1).f(b) < 0$ then approximate root lies in $[x, b]$

else if, $f(a).f(x_1) < 0$ then let $b = x_1$ approximate root lies in $[a, x_1]$

Step 4. Repeat steps 2 and 3 until desired accuracy is obtained.

Example 1. Find a root of an equation $f(x) = x^2 - 3$ using Regula Falsi Method up to four iterations.

Solution:

here $f(x) = x^2 - 3$

so, $f(1) = -2 < 0$ and $f(2) = 1 > 0$

therefore, root lies between $a = 1$ and $b = 2$

since $f(1)f(2) < 0$

1st iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{(1)(1) - 2(-2)}{1 - (-2)} = \frac{1+4}{3} = 1.6667$$

$$f(x_1) = f(1.6667) = (1.6667)^2 - 3 = -0.2222 < 0$$

2*nd* iteration:

Here $f(1.6667) = -0.2222 < 0$ and $f(2) = 1 > 0$

root lies between $a = 1.6667$ and $b = 2$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_2 = \frac{(1.6667)(1) - 2(-0.2222)}{1 - (-0.2222)} = 1.7273$$

$$f(x_2) = f(1.7272) = (1.7273)^2 - 3 = -0.0165 < 0$$

3*rd* iteration:

Here $f(1.7273) = -0.0165 < 0$ and $f(2) = 1 > 0$

Root lies between $a = 1.7273$ and $b = 2$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_3 = \frac{(1.7273)(1) - 2(-0.0156)}{1 - (-0.0156)} = 1.7317$$

$$f(x_3) = f(1.7317) = (1.7317)^2 - 3 = -0.0012 < 0$$

4*th* iteration:

Here $f(1.7317) = -0.0012 < 0$ and $f(2) = 1 > 0$

root lies between $a = 1.7317$ and $b = 2$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_4 = \frac{(1.7317)(1) - 2(-0.0012)}{1 - (-0.0012)} = 1.732$$

$$f(x_4) = f(1.732) = (1.732)^2 - 3 = -0.000176 < 0$$

Approximate root of the equation $x^2 - 3 = 0$ using False Position method is 1.732 (After 4 iterations).

Example 2. Find a root of the equation $2e^{x}\sin x = 3$ using the false position method and correct it up to two decimal places.

Solution:

Let $f(x) = 2e^{x}\sin x - 3 = 0$

so, $f(0) = 2e^{0}\sin 0 - 3 = 0$

$f(0) = -3 < 0$

also $f(1) = 2e^{1}\sin(1) - 3$

$f(1) = 1.574770$

since $f(0)f(1) < 0$

therefore, root lies between 0 and 1.

First Iteration

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

here $a = 0$ and $b = 1$

$$x_1 = \frac{0 \times (1.5747) - 1 \times (-3)}{1.5747 + 3}$$

$$x_1 = 0.6557$$

Now $f(x_1) = f(0.6557) = 2e^{0.6557}\sin(0.6557) - 3$

$f(x_1) = -0.6507 < 0$

since $f(0.6557f)f(1) < 0$

Therefore, roots lie between 0.6557 and 1.

Second Iteration

$a = 0.6557$ and $b = 1$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_2 = \frac{0.6557(1.5747) - (0.6557)(-0.6507)}{1.5747 - (-0.6507)}$$

$$x_2 = 0.7563$$

Now $f(x_2) = f(0.7563) = -0.0761 < 0$

since $f(0.7563)f(1) < 0$

therefore, roots lie between 0.7563 and 1.

Third Iteration

$$x_3 = \frac{(0.7563)(1.5747) - 1(-0.0761)}{1.5747 - (0.0761)}$$

$$x_3 = 0.7675$$

Then the best approximation of the roots up to two decimal places is 0.768.

Exercise 12.2

Use Regula Falsi method to find a root of following equations:

1. $x^3 - 2x - 5 = 0, [2, 3]$ up to three iterations.
2. $\sin(2x) - e^{x-1} = 0, [-2, -1]$ up to four iterations.
3. $x^4 - x - 10 = 0, [1,2]$ up to two decimal places.
4. $3x + \sin(x) - e^x = 0, [0,1]$ up to three decimal places.
5. $f(x) = 2\cos x - x = 0, [1, 2]$ up to five decimal places.

- **Newton's Raphson method**

The Newton Raphson Method is also commonly known as Newton's Method. It is an iterative procedure for determining a better approximation for the root of a continuous, differentiable function $f(x) = 0$ at $x = x_0$

Algorithm

Step 1.

Let $f(x)$ is differentiable function over (a, b) then to find the approximate root of $f(x) = 0$ by Newton's Rephson method, we have to take initial guess $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Following steps are taken to find approximate root by bisection method.

Step 2.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step 3.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Step 4.

By continuing this process, desired accuracy is obtained.

Example 1. Find a root of an equation $x^2 - 3 = 0$, using Newton Raphson method up to three iteration.

Solution:

Let $f(x) = x^2 - 3 = 0$

$f'(x) = 2x$

For our simplicity, we take initial guess $x_0 = 1.5$

1*st* iteration:

$f(x_0) = f(1.5) = (1.5)^2 - 3 = -0.75$

$f'(1.5) = 2(1.5) = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \frac{-0.75}{3} = 1.75$$

2*nd* iteration:

$$f(x_1) = f(1.75) = (1.75)^2 - 3 = 0.0625$$

$$f'(1.75) = 2(1.75) = 3.5$$

$$x_2 = x_2 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 1.75 - \frac{0.0625}{3.5} = 1.7321$$

3*rd* iteration:

$$f(x_2) = f(1.7321) = (1.7321)^2 - 3 = 0.0003$$

$$f'(1.7321) = 2(1.7321) = 3.4643$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_2 = 1.7321 - \frac{0.0003}{3.4643} = 1.7321$$

Approximate root of the equation $x^2 - 3 = 0$, using Newton's method is 1.7321 (After 3 iterations).

Example 2. Find a root of $3x - \cos x - 1 = 0$ by Newton's Raphson method, correct up to 4 decimal places.

Solution:

Let $f(x) = 3x - \cos x - 1 = 0$

$f'(x) = 3 + \sin x$

We take initial guess $x_0 = 0$

Here $f(0) = 3(0) - \cos 0 - 1$

$f(0) = -2$

and $f'(0) = 3 + \sin(0)$

$f'(0) = 3$

1st iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{(-2)}{3}$$

$$x_1 = 0.6667$$

2*nd* iteration:

$$f(x_1) = f(0.6667) = 3(0.6667) - \cos 0.6667 - 1$$
$$f(x_1) = 0.000167$$
$$f(x_1) = 3 + \sin(0.6667)$$
$$f(x_1) = 3.01163$$

Now

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = 0.6667 - \frac{0.000167}{3.01163}$$
$$x_2 = 0.6667 - 0.0000554$$
$$x_2 = 0.6666$$
$$f(0.6666) = 3(0.6666) - \cos(0.6666) - 1$$
$$f(0.6666) = 0.000032$$

Hence 0.6666 is the approximate root correct up to four decimal places.

Exercise 12.3

Use Newton Raphson method to find a real root of following functions:

1. $f(x) = 2x^3 - 2x - 5$ up to three iterations with initial guess $x_0 = 2$.
2. $f(x) = x^3 - x - 1$ up to three iterations with initial guess $x_0 = 1$.
3. $f(x) = x^3 - 2x - 5$ up to two iterations with initial guess $x_0 = 1$.
4. $f(x) = 2\cos x - x, x_0 = 0$
5. $f(x) = 2^x - x - 1.7, x_0 = 1.5$
6. $f(x) = 3x - e^x, x_0 = 0$
7. $f(x) = 3x - \sqrt{1 + \sin x}, x_0 = 1$

12.2 Numerical Quadrature

Quadrature refers to any method for numerically approximating the value of a definite integral

$$\int_a^b f(x)dx.$$

The estimated calculation of an integral using numerical technique is known as numerical integration.

12.2.1 Define numerical quadrature. Use:

- **Trapezoidal rule,**
- **Simpson's rule, to compute the approximate value of definite integrals without error terms.**

Trapezoidal Rule

The Trapezoidal rule is an integration rule that evaluates the area under the curve by dividing the total area into smaller trapezoids rather than using rectangles.

Trapezoidal Rule Formula

To prove the trapezoidal rule, consider a curve as shown in the figure 12.1 above and divide the area under that curve into trapezoids.

Let $f(x)$ be a continuous function on the interval $[a,b]$. Now divide the interval $[a,b]$ into n sub intervals with each of equal width Δx.

We use here formula to calculate the width $\Delta x = h$ of each subinterval.

$\Delta x = h = \frac{b-a}{n}$ such that $a = x_0 < x_1 < x_2 < x_3 <, \ldots, < x_n = b$.

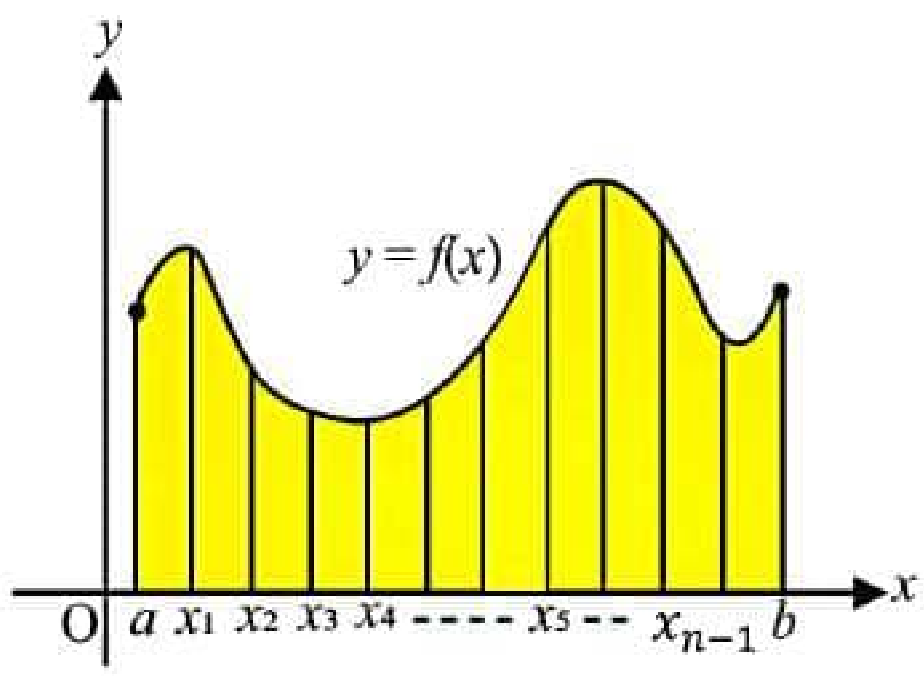


Fig. 12.1

We see that the first trapezoid has a height Δx and length of parallel base is the sum are $f(x_0)$ or y_0 and $f(x_1)$ or y_1 respectively.

Thus, the area of the first trapezoid in the above figure can be given as,

$$\frac{1}{2}\Delta x[f(x_0) + f(x_1)]$$

The areas of the remaining trapezoids are

$$\frac{1}{2}\Delta x[f(x_1) + f(x_2)], \frac{1}{2}\Delta x[f(x_2) + f(x_3)]$$

and so on.

Consequently,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + f(x_1)) + \frac{\Delta x}{2}(f(x_1) + f(x_2)) + f(x_0) +$$
$$\frac{\Delta x}{2}(f(x_2) + f(x_3)) + \ldots + \frac{\Delta x}{2}(f(x_{n-1}) + f(x_n))$$

After taking out a common factor of $\frac{\Delta x}{2}$ and combining like terms, we have,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2(f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1})) + f(x_n)]$$

Then the Trapezoidal Rule formula for area approximating the definite integral $\int_a^b f(x)dx$ is given by:

If we take $y = f(x)$ and $\Delta x = h$ then,

$$\int_a^b ydx \approx T_n = \frac{h}{2}[y_0 + 2(y_1 + 2y_2 + \ldots + 2y_{n-1}) + y_n]$$

Example 1. Approximate the area under the curve $y = f(x)$ between $x = 0$ and $x = 8$ using Trapezoidal Rule with $n = 4$ subintervals. A function $f(x)$ and x values are given in the following table:

x	0	2	4	6	8
$f(x)$	3	7	11	9	3

Solution: As $f(x)$ and x are given in the table:

The Trapezoidal Rule formula for $n = 4$ subintervals is given as:

$$T_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$T_4 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)]$$

Here the subinterval width $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

Now, substitute the values from the table to find the approximate value of the area under the curve.

$$A \approx T_4 = \frac{2}{2}[3 + 2(7 + 11 + 9) + 3]$$

$$A \approx T_4 = [6 + 2(27)]$$

$$A \approx T_4 = 60 \text{ unit square.}$$

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 60.

Example 2. Evaluate $\int_2^7 \frac{1}{x}dx$ using Trapezoidal rule, taking subintervals $n = 5$.

Solution:

$$f(x) = \frac{1}{x}$$

Here $a = 2$ and $b = 7$, subintervals $n = 5$

Now, $$h = \frac{b-a}{n} = \frac{7-2}{5} = 1$$

The values of x and $f(x)$ are given in the following table:

x	2	3	4	5	6	7
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

OR

x	2	3	4	5	6	7
$f(x)$	0.5	0.3333	0.25	0.2	0.1667	0.1429

Using Trapezoidal Rule

$$T_5 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3) + f(x_4)) + f(x_5)]$$

$$T_5 = \frac{1}{2}[0.5 + 2(0.3333 + 0.25 + 0.2 + 0.1667) + 0.1429]$$

$$T_5 = \frac{1}{2}[0.5 + 2(0.95) + 0.1429]$$

$$T_5 = 1.2714$$

Example 3. Evaluate $\int_1^2 e^{\frac{1}{x}}\,dx$ using Trapezoidal rule with subintervals $n = 6$.

Solution:

$$f(x) = \int_1^2 e^{\frac{1}{x}}\,dx$$

Here $a = 1$ and $b = 2$, subintervals $n = 6$

Now, $h = \frac{b-a}{n} = \frac{2-1}{6} = 0.1667$

The values of x and $f(x)$ are given in the following table:

x	1	1.1667	1.3333	1.5	1.6667	1.8333	2
$f(x)$	2.7183	2.3564	2.117	1.9477	1.8221	1.7254	1.6487

Using Trapezoidal Rule

$$T_6 = \frac{h}{2}[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) + f(x_6)]$$

$$T_6 = \frac{0.1667}{2}[2.7183 + 2(2.3564 + 2.117 + 1.9477 + 1.8221 + 1.7254) + 1.6487]$$

$$T_6 = \frac{0.1667}{2}[2.7183 + 2(0.99687) + 1.6487] = 2.0254 \text{ unit square.}$$

Simpson's Rule

Simpson's rule is an extension of Trapezoid rule. It contains two different schemes Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. Here, we discuss each one separately.

Simpson's $\frac{1}{3}$ Rule

Let $f(x)$ be a continuous function on $[a, b]$, then value of $\int_a^b f(x)\,dx$ by Simpson's $\frac{1}{3}$ rule is calculated by

$$\text{Simpson's } \frac{1}{3} = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\right]$$

Here h is the width of each interval and calculated by $h = \frac{b-a}{n}$

Where n denotes number of subintervals.

Note: The method is only valid if n is the multiple of 2.

Example 1. Evaluate $\int_0^2 2x\,dx$ using Simpson's 1/3 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^2 2x\,dx$,

Take $a = 0, b = 2$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3} = 0.3333$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$
$f(x)$	0	0.6667	1.3333	2	2.6667	3.3333	4

Use Simpson's 1/3 rule by taking $y = f(x)$ for $n = 6$

$$S\frac{1}{3} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$S\frac{1}{3} = \frac{0.3333}{3}[(0 + 4) + 4(0.6667 + 2 + 3.3333) + 2(1.3333 + 2.6667)]$$

$$S\frac{1}{3} = \frac{0.3333}{3}[(0 + 4) + 4(6) + 2(4)] = 4 \text{ square unit.}$$

Example 2. Evaluate $\int_0^1 e^x\,dx$ by Simpson's 1/3 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^1 e^x\,dx$,

Take $a = 0, b = 1$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = 0.667$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
$f(x)$	1	1.1814	1.3956	1.6487	1.9477	2.301	2.7183

Use Simpson's 1/3 rule by taking $y = f(x)$

$$S\frac{1}{3} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$S\frac{1}{3} = \frac{0.1667}{3}[(1 + 2.7183) + 4(1.1814 + 1.6487 + 2.301) + 2(1.3956 + 1.9477)]$$

$$S\frac{1}{3} = \frac{0.1667}{3}[(1 + 2.7183) + 4(5.1311) + 2(3.3433)]$$

$$S\frac{1}{3} = 1.7183 \text{ square units.}$$

Example 3. Evaluate $\int_1^2 e^{x^3}\,dx$ using Simpson's 1/3 rule by taking $n = 4$.

Solution:

Let $f(x) = \int_1^2 e^{x^3}\,dx$

Take $a = 1, b = 2$, and $n = 4$.

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

The values of $f(x)$ at x are given in the following table:

x	1	1.25	1.5	1.75	2
$f(x)$	2.7183	7.0507	29.2243	212.592	2980.958

Use **Simpson's 1/3 rule**

$$S\frac{3}{8} = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$S\frac{3}{8} = \frac{0.25}{3}[(2.7183 + 2980.958) + 4(7.0507 + 212.592) + 2(29.2243)]$$

$$S\frac{3}{8} = \frac{0.25}{3}[(2.7183 + 2980.958) + 4(219.6427) + 2(29.2243)]$$

$$S\frac{3}{8} = 326.7246 \text{ square units.}$$

Simpson's $\frac{3}{8}$ Rule

Let $f(x)$ be a continuous function on $[a, b]$ there values of $\int_a^b f(x)$ by Simpson's $\frac{3}{8}$ rule is calculated by

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

Here h is the length of each interval and calculated by $h = \frac{b-a}{n}$

Where n is the number of subintervals. The formula is used when n is the multiple of 3.

Example 1. Evaluate $\int_0^1 e^x\,dx$ by Simpson's 3/8 rule with $n = 6$.

Solution:

Let $f(x) = \int_0^1 e^x\,dx$,

Take $a = 0, b = 1$, and $n = 6$.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = 0.667$$

The values of $f(x)$ at x are given in the following table:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$
$f(x)$	1	1.1814	1.3956	1.6487	1.9477	2.301	2.7183

Use Simpson's 1/3 rule by taking $y = f(x)$

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$S\frac{3}{8} = \frac{3(0.1667)}{8}[(1 + 2.7183) + 2(1.6487) + 3(1.1814 + 1.3956 + 1.9477 + 2.301)]$$

$$S\frac{3}{8} = \frac{3(0.1667)}{8}[(1 + 2.7183) + 2(1.6487) + 3(6.8257)]$$

$$S\frac{3}{8} = 1.7183 \text{ unit square}$$

Example 2. Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by Simpson's 3/8 rule with $n = 6$.

Solution:

$$f(x) = \int_0^6 \frac{1}{1+x^2}dx$$

Here $a = 0$ and $b = 6$, Subintervals $n = 6$

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

The values of x and $f(x)$ are given in the following table:

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027

Using Simpson's 3/8 Rule

$$S\frac{3}{8} = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$S\frac{3}{8} = \frac{3 \times 1}{8}[(1 + 0.027) + 2(0.1) + 3(0.5 + 0.2 + 0.0588 + 0.0385)]$$

$$S\frac{3}{8} = \frac{3 \times 1}{8}[(1 + 0.027) + 2(0.1) + 3(0.7973)]$$

$$S\frac{3}{8} = 1.3571 \text{ unit space}$$

12.2.2 Use MAPLE command Trapezoid for trapezoidal rule and SIMPSON for Simpson rule and demonstrate through examples

The trapezoidal rule to compute an approximation to a definite integral. The call trapezoid $(f(x), x, n)$ finds an approximation to the definite integral $\int_a^b f(x)dx$ using n subdivisions of the interval $[a, b]$. We use the trapezoidal rule with $n = 12$ to find an approximation to $\int_a^b \frac{1}{\sqrt{1+x^4}} dx$,

where, $f(x)$ stands for an algebraic expression in x
x variable of integration
a lower bound on the range of integration
b upper bound on the range of integration
n stands for the number of trapezoids to use(optional)

Note: All above operators should be taken from the Maple calculus pallets.

Trapezoidal Rule

$> with(Student[Calculus1])$:

$> ApproximateInt\left(\frac{1}{\sqrt{1+(x)^4}}, x = 0..1, method = trapezoid, output = plot\right);$

An approximation of $\int_0^1 f(x)\,dx$ using Trapezoid rule, where $f(x) = \frac{1}{\sqrt{x^4+1}}$ and the partition is uniform. The approximate value of the integral is 0.9264474916. Number of subintervals used: 10.

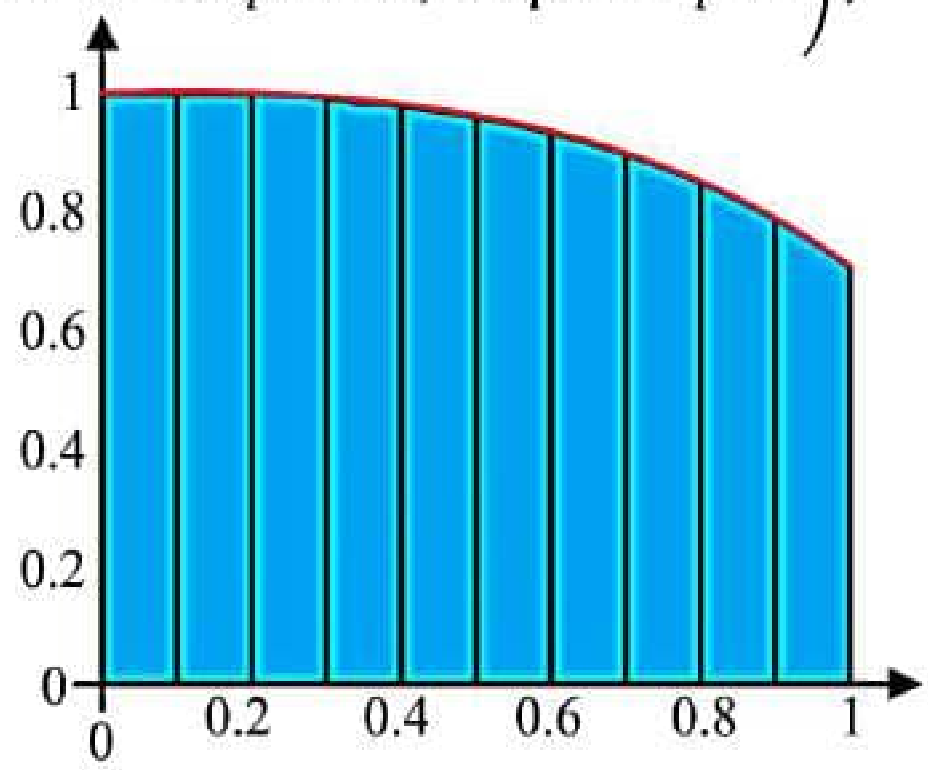


Fig. 12.2

$> ApproximateInt$
$\left(\begin{array}{c}1 + exp(x), x = 0..3, method = trapezoid,\\ output = plot, partition = 10\end{array}\right)$

An approximation of $\int_0^3 f(x)\,dx$ using Trapezoid rule, where $f(x) = 1 + e^x$ and the partition is uniform. The approximate value of the integral is 22.22846420. Number of subintervals used: 10.

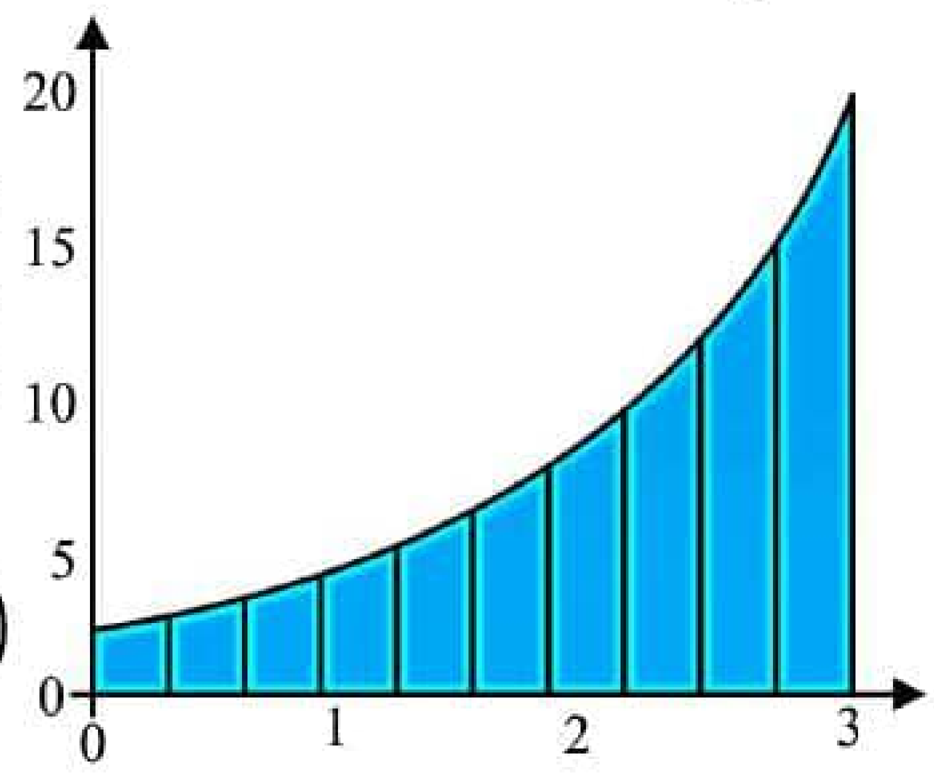


Fig. 12.3

$> ApproximateInt$
$\left(\begin{array}{c}cos(x) - exp(-x), x = 0.5..3.5, method = trapezoid,\\ output = plot, partition = 10\end{array}\right)$

An approximation of

$$\int_{0.5}^{3.5} f(x)\,dx$$

using Trapezoid rule, where

$$f(x) = \cos(x) - e^{-x}$$

and the partition is uniform. The approximate value of the integral is -1.404622147. Number of subintervals used: 10.

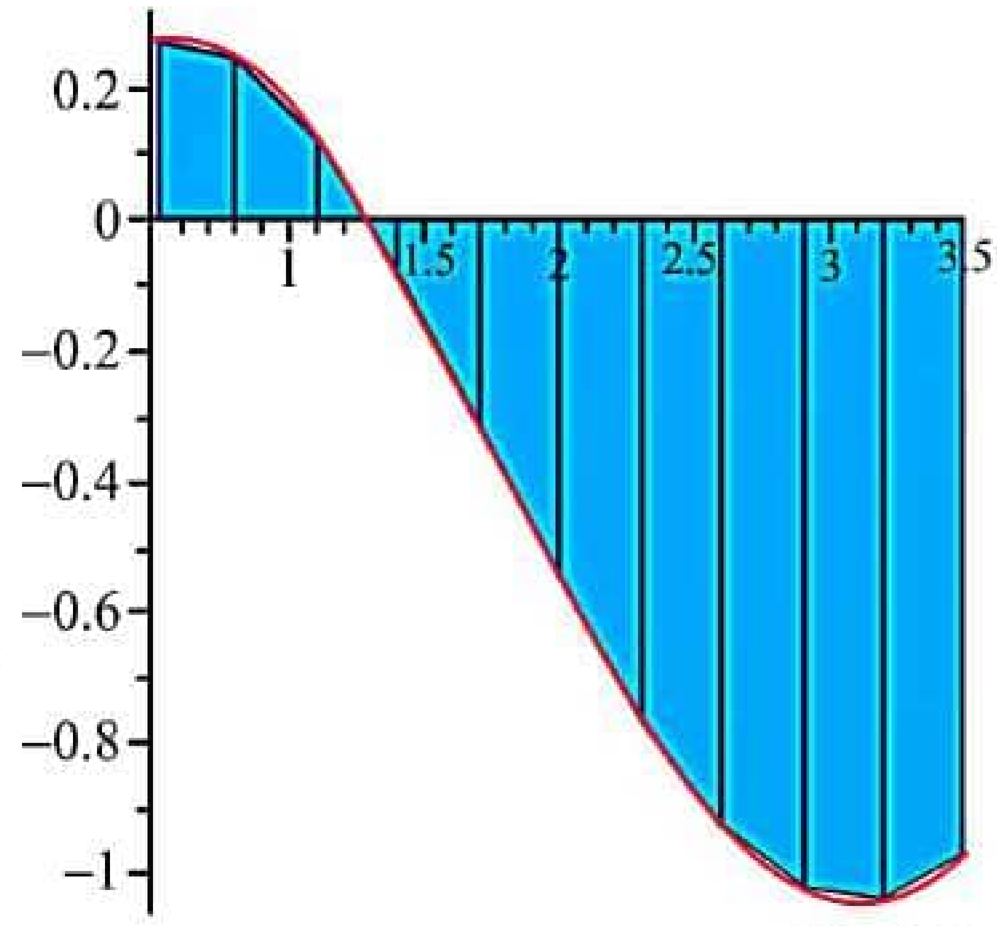


Fig. 12.4

Simpson Rule

$> ApproximateInt$
$\left(1/(x^2 + 3 * x + 2), x = 0..3, method = simpson, output = plot, partition = 10\right)$

An approximation of $\int_0^3 f(x)\,dx$ using Simpson's rule, where $f(x) = \frac{1}{x^2+3x+2}$ and the partition is uniform. The approximate value of the integral is 0.4700185982. Number of subintervals used: 10.

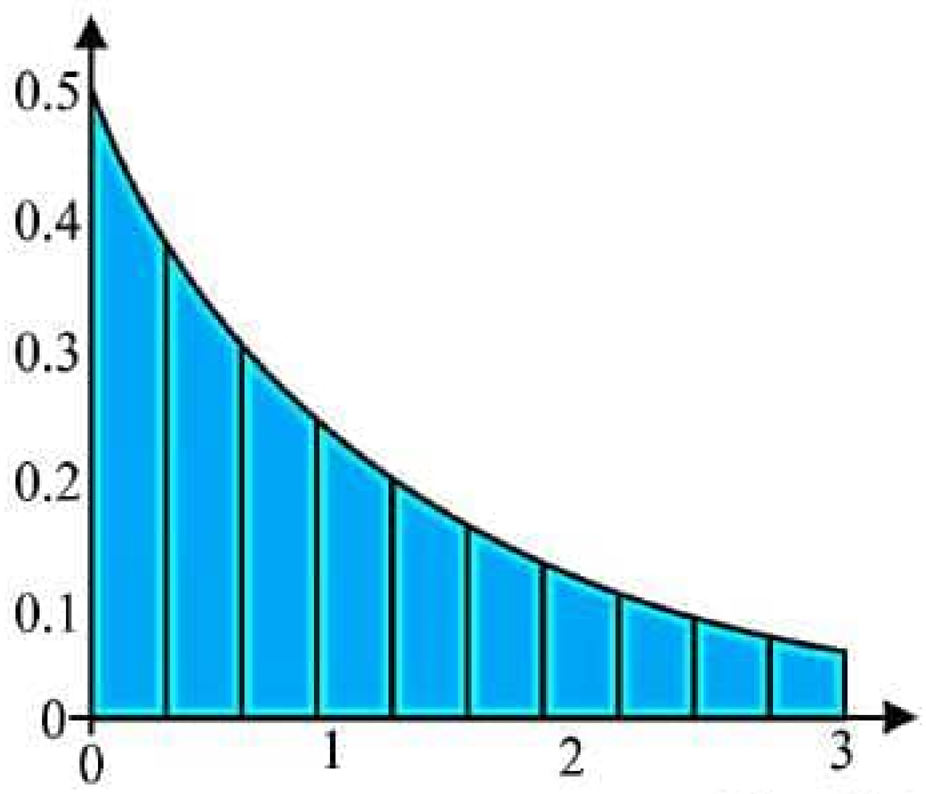


Fig. 12.5

$> ApproximateInt$
$\left(exp(x^2), x = -1..1, method = simpson, output = plot, partition = 10\right)$

An approximation of $\int_{-1}^{1} f(x)\,dx$ using Simpson's rule, where $f(x) = e^{x^2}$ and the partition is uniform. The approximate value of the integral is 2.925362800. Number of subintervals used: 10.

Note: Before executing above commands, it is important to write with (student [calculus1]).

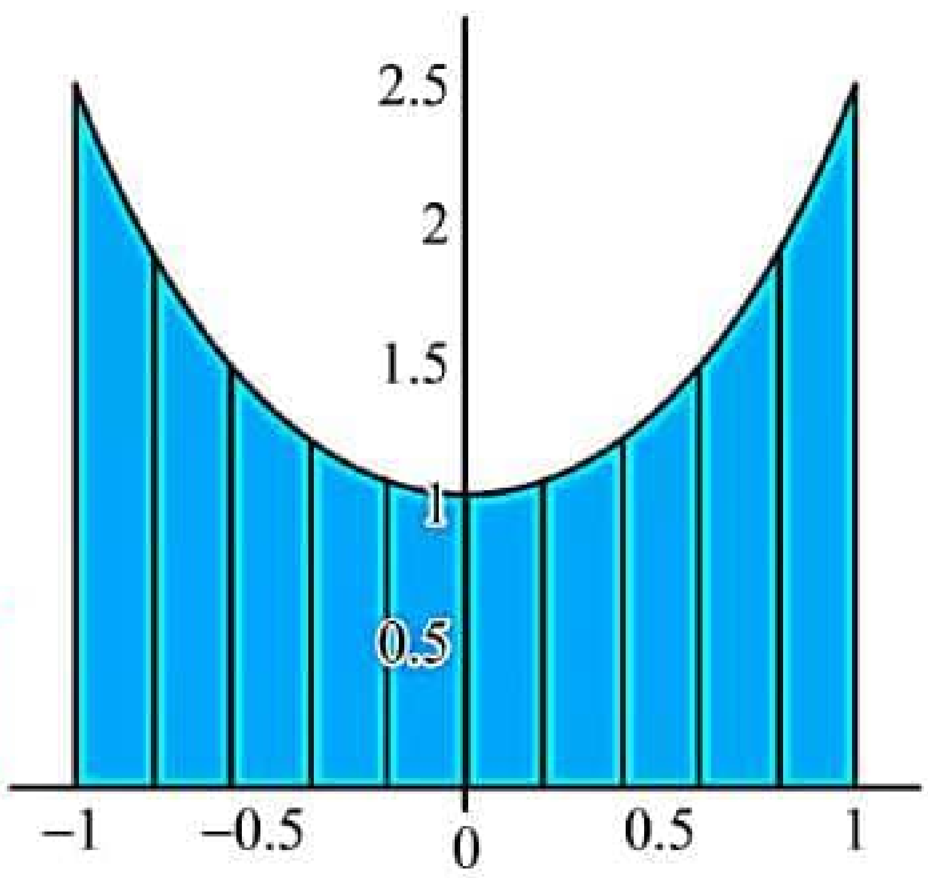


Fig. 12.6

Exercise 12.4

1. Evaluate the following integrals by Trapezoidal rule

(i) $\int_0^2 e^x\,dx$ with 6 intervals (ii) $\int_1^3 \frac{1}{\sqrt{x}}dx$ with 5 intervals

(iii) $\int_0^{\frac{\pi}{2}} \sin x\,dx$ with 7 intervals (iv) $\int_0^{1.4} e^{-x^2}dx$ with 5 intervals

2. Evaluate the following integrals by Simpson $\frac{1}{3}$ rule

(i) $\int_2^3 (4x^2+6)dx$, with 4 intervals (ii) $\int_0^2 \frac{1}{e^x}dx$, with 6 intervals

(iii) $\int_0^{\frac{\pi}{3}} \sqrt{\sin x}\,dx$ with 6 intervals (iv) $\int_0^1 \frac{2}{1+x^2}dx$ with 8 intervals

3. Evaluate the following integrals by Simpson $\frac{3}{8}$ rule

(i) $\int_1^3 \sqrt{x}\,dx$, with 6 intervals (ii) $\int_1^6 \frac{\ln x}{x}dx$, with three sub intervals

(iii) $\int_0^2 \frac{e^{2x}}{1+x^2}dx$ with 9 sub intervals (iv) $\int_0^{\frac{\pi}{4}} \sin x\,dx$ with 6 sub intervals

4. Write MAPLE Command Trapezoid for trapezoidal rule and SIMPSON for Simpson rule

(i) $\frac{x^2-2}{2}$, $x = 0..1$ $n = 10$, method trapezoiadal rule

(ii) $\sqrt{9+x^2}$, $x = 0..4$ $n = 10$, method trapezoiadal rule

(iii) $\frac{1}{x^2+4x+3}$, , $x = 0..2$ $n = 10$, method simpson's rule

(iv) e^{-x^2} , $x = 0..2$ $n = 10$, method simpson's rule

Review Exercise 12

1. Select the correct option.

(i) If real root of an equation $f(x) = 0$ lies in the interval $[a, b]$ then $f(a)f(b)$ will be

(a) > 0 (b) < 0 (c) $= 0$ (d) All of them

(ii) In bisection method, the approximate root is a/an __________ of end point of an interval in which actual root lies

(a) Arithmetic mean (b) Geometric mean (c) Sum (d) Product

(iii) Iterative formula for False Position Method to solve the equation $f(x) = 0$ at interval $[a, b]$ is

(a) $\frac{af(a)-bf(b)}{f(a)-f(b)}$ (b) $\frac{af(b)-bf(a)}{a-b}$

(c) $\frac{af(a)-bf(b)}{f(b)-f(a)}$ (d) $\frac{af(b)-bf(a)}{f(b)-f(a)}$

(iv) The fastest method to solve the nonlinear equation numerically is
(a) Bisection Method (b) False Position Method
(c) Newton Raphson Method (d) Both a and b

(v) Newton Raphson Method fails when derivative value of $f(x)$ becomes
(a) > 0 (b) < 0 (c) $= 0$ (d) All of them

(vi) Iterative formula of Newton Raphson method of solve $f(x) = 0$ is ______________
(a) $x_{n+1} = x_n + \frac{f'(x_n)}{f(x_n)}$ (b) $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$
(c) $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$ (d) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(vii) Numerical integration comprises a broad family of algorithms for calculating the numerical value of a?
(a) Definite integral (b) Indefinite integral
(c) Simple integral (d) Compound integral

(viii) In Trapezoidal rule the number of sub interval is the multiple of:
(a) 0 (b) 1 (c) 2 (d) 3

(ix) In Simpson One Third Method, the number of subinterval is the multiple of
(a) 4 (b) 1 (c) 2 (d) 3

(x) The fastest method to solve the definite integral numerically is
(a) Trapezoidal Rule (b) Simpson One Third Rule
(c) Simpson Three Eight Rule (d) Both a and b

2. Using Bisection method find the root of $cosx - xe^x$ with a = 0 and b = 1, by taking 5 iterations.
3. Find a root for the equation $2e^x sinx = 1$ using the false position method and correct it to three decimal places with three iterations, taking $[1, 2]$ as an interval.
4. Find the cube root of 12 using the Newton Raphson method assuming $x_0 = 2.5$.
5. Solve $\int_0^1 cosx^2 dx$ using trapezoidal rule for $n = 5$.
6. Solve $\int_{-2}^4 e^{x^2} dx$ using Simpson one Third as well as Simpson Three Eight rule for $n = 6$.